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TWO-DIMENSIONAL FRACTAL LINEARIZATION OF DISTRIBUTION CURVES

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Distribution curves of properties of materials (size, density, hydrophobicity, etc.) are important for characterization and controlling separation results. Frequently, the mass-based size distribution curves are linearized using various functions including those of Rosin-Rammler, Gates-Gaudin-Schumann, and Gaudin-Meloy. In this paper, a fractal approach was tested for linearization of the size distribution curves. It was shown in the paper that the three-dimensional (3D) fractal linearization equation is the same as the Gates-Gaudin-Schumann formula. It was also shown that area-based 2D fractal can be used for linearization of the size distribution curves provided that an appropriate area, on which the sample is spread, is determined. It was also shown that in some cases more than one fractal is necessary for linearization of the size distribution curve.

Key words: fractal geometry, particle size distribution, linearization

INTRODUCTION

Knowledge of distribution of particulate material properties such as density, size, hydrophobicity, magnetic susceptibility, etc., is important in many industrial applications including cement, food, pharmaceuticals, cosmetics, pigment, fertilizers, and mineral processing. The distribution of the properties is usually plotted in a graphical form as population or equivalent quantity (cumulative or non-cumulative) versus the value of the feature of particles. It is also popular to make the distribution plots linear. Different mathematical formulas and approaches have been suggested to make linear the distribution curves, especially the size distribution. Selected formulas used for linearization of particle size distribution are given in Table 1. The formulas may contain one, two or more fitting parameters.

Another approach to make the size distribution curve a line is the use of fractals. The fractal approach has been successfully applied to describe irregular particles

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(Chermant and Coster, 1978; Flook, 1978; Whalley and Orford, 1982; Kaye, 1981; 1984), adsorption phenomena on irregular surfaces (Nowak, 1993), impedance response of rough electrodes (Nowak, 1993), rock porosity and fracture of surfaces (Kaye, 1984), bioprocessing of ores (Kaye,1985), computer simulation of flocs (Tang, 2002), and others.

Table 1. Selected functions applied for linearization of size distribution curves. Symbols n, s, and S stand for constants (Kelly and Spottiswood, 1982; Drzymala, 2001)

Name	$\Sigma\lambda$ (%) = (cumulative mass fraction (or percent) passing a given d)	Meaning of d^*
Rosin–Rammler or Weibull	$1 - \exp[-(d/d^*)^s]$	d value at $\Sigma\lambda = 0.632$
Gates–Gaudin–Schumann	$[d/d^*]^n$	maximum d value
Broadbent–Callcott	$1 - \exp[-(d/d^*)]/(1 - \exp(-1)$	maximum d value
Gaudin–Meloy	$1 - [1 - (d/d^*)]^n$	maximum d value
Log-probability	$\mathrm{erf}\,[\ln(d/d^*)/\sigma]$, erf – error function, σ – geometric standard deviation	median d value
Harris	$1 - [1 - (d/d^*)^S]^n$	maximum d value

Tyler and Wheatcraft (1992) proposed application of three-dimensional (3D) fractals for linearization of mass-based size distribution curves. They developed normalized function relating mass (weight) of a size fraction of particles with the sieve diameters and 3D-fractal dimension (D_3) as follows:

$$\frac{M(D<d)}{M_t} = \left(\frac{d}{d_{\max}}\right)^{(3-D_3)} \quad (1)$$

where $M(D<d)$ is the cumulative mass of particles having size D smaller than a considered comparative sieve size d, M_t is the total mass of sample (for normalization), and d_{max} is the maximum screen size. According to Tyler and Wheatcraft (1992) a plot of $\frac{M(D<d)}{M_t}$ against $\left(\frac{d}{d_{\max}}\right)$ in a log-log form yields a linear plot with 3D-fractal dimension D_3, related to the line slope coefficient n, where $n=3-D_3$. Applying Eq. 1 for any mass-based size distribution data, 3D-fractal dimension D_3 can be determined. A comparison of the 3D-fractal function given by Eq. 1 with other traditional linearization equations shown in Table 1 indicates that Eq. 1 is the same as the Gates-Gaudin-Schuman (Eq. 2 in Table 1) formula traditionally

used for the linerarization. This indicates that the Gates-Gaudin-Schuman equation is a fractal and the slope (n) of the Gates-Gaudin-Schuman line equals 3-D_3.

Another approach to make linear size distribution curves using the fractal concept was proposed by Hargrave et. al. (1998) and is based on a 2D fractal. They used scanning image analyses to express air bubbles distribution in flotation froth using a general equation:

$$\beta = c\alpha^{(2-D_2)} \tag{2}$$

where β is the normalized area of bubbles having diameters less than a given normalized size and which is calculated from:

$$\beta = 1 - \frac{area\ of\ bubbles\ having\ diameters > normalized\ diameter, \alpha}{total\ area\ of\ field\ of\ view\ (area\ of\ the\ top\ view\ of\ the\ flotation\ cell\ )} \tag{3}$$

where C is a constant, D_2 is the 2D-fractal dimension, and α is the normalized bubble size determined by dividing a given bubble diameter (d) by the maximum bubble diameter (d_{max}).

Having 2D and 3D analyses of particles is not always possible to accurately linearize the size distribution. In the case of the 3D-fractal analysis, sometimes two or more curves are produced while the results of linearization of particles size distribution with the 2D-fractal analysis depend on the area on which a sample of the particles is spread for analysis. Therefore, the goal of this study is to work out a procedure for 2D-fractal linearization of the distribution curves.

TRADITIONAL VERSUS 2D FRACTAL GEOMETRY

In classical geometry, the integer dimensions 0, 1, 2, and 3 are known to express point, line, area, and volume, respectively. Thus, a dimension of an object is the number of coordinates required to express the position of a certain point located in this object. In fractal geometry of Mandelbrot (1977, 1983), it is assumed that the dimensions does not have to be integers. They can be any real value from 0 to 3. The fractal dimension improves the description of irregular objects. For example, lines can be described by fractal geometry to have dimension of 1+x where $0 \leq x < 1$, and x = 0 for straight lines, and any value in-between for zigzag lines. At the same time, areas of uniform shapes have a Euclidean dimension of two, while they can assume values between one and two considering the fractal concept. This is also valid for objects occupying space. The Euclidean dimension is usually three while their fractal dimension can be in the range from two to three. Figure 1 shows Euclidean versus fractal dimensions for line, area and volume of objects.

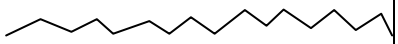
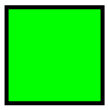
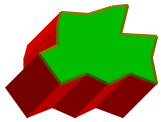
	a	b
Linear entities	Fractal dimension =1 Traditional dimension =1	 Fractal dimension =1.15 Traditional dimension =1
Area entities	 Fractal dimension =2 Traditional dimension =2	 Fractal dimension =1.90 Traditional dimension =2
Volume entities	 Fractal dimension =3 Traditional dimension =3	 Fractal dimension =2.65 Traditional dimension =3

Fig. 1. Fractal versus Euclidean dimensions expressing line, area, and volume objects

There are various methods to determine the fractal dimension. The basic concept used in all techniques is based on a power-law relationship. Such techniques include the parallel-line (Hyslip and Vallejo, 1997), divider (Hyslip and Vallejo, 1997), random walk (Kaye, 1995), in addition to the Sierpinski carpet method (Kaye, 1988; Hargrave et al., 1998). The later one depends on creating similar shapes (triangles, squares, circles etc.) called fractals packed in a given space, called the Sierpinski carpet, according to a certain rule. Figure 2 shows the Sierpinski carpet having the constructor (main carpet) (a), generator (shape and size of the first entity) (b), and first (c) and second (d) iteration algorithms. The mathematical form to express the area-based (2D) fractal dimension for the Sierpinski carpet is given by:

$$D_2=\ln(N)/\ln(1/r) \tag{4}$$

where D_2 is the 2D-fractal dimension, N= the total number of the un-removed squares, usually constant for each iteration step (in this case N = 8) and r = linear ratio of similarity between repeated shapes present in two subsequent iterations (in this case r = 1/3). From a theoretical point of view, for ideal systems, both N and r can be evaluated, and thus the fractal dimension D_2 can be calculated. For real objects, the determination of both N and r is ambiguous arising from the difficulties to find fractal dimension D_2. Therefore, there is a need for explicit mathematical steps to make it easier to calculate fractal dimension D_2.

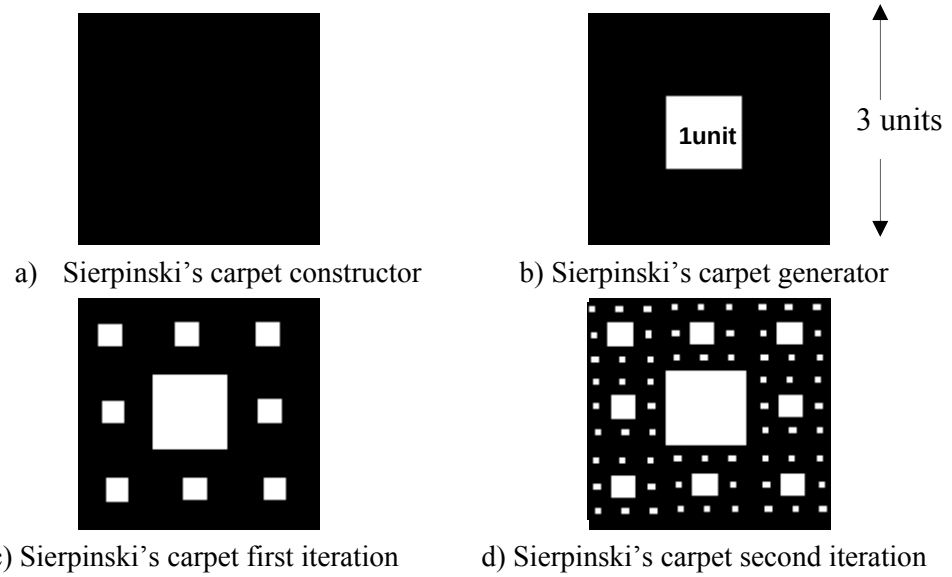


Fig. 2. The Sierpinski carpet showing up to the second iteration order having fractal dimension of 1.8928 (after Hargrave et al., 1998)

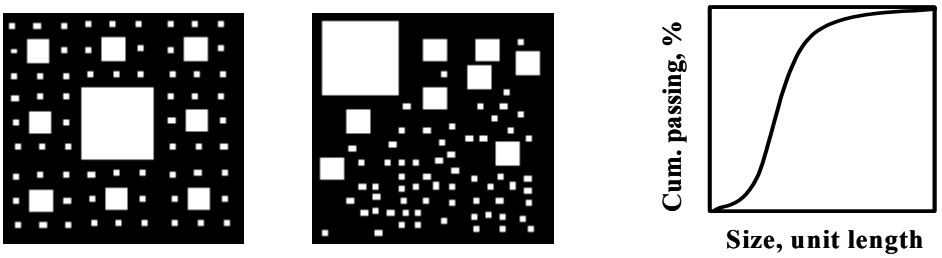
Further iterations on the Sierpinski carpet shown in Fig. 2 can lead to infinite number of squares with different sizes causing the carpet area to vanish. By considering such squares (objects that may also have any shape) and "shaking the carpet" one gets objects with different projected areas (Fig. 3) spread over the base of the Serpinski carpet (Fig.3b). Therefore, the Sierpinski carpet can be used to represent area-based size distribution. The correlation between the size distribution and 2D-fractal dimension represented by the Sierpinski carpet is similar to Hargrave et al., (1998) approach for calculation of air bubbles distribution in a given froth. Except that the area of the top view of the flotation cell is constant providing constant background. In the case of particles, the background (the field over which the sample of particles was spread) it can be any area. In Eq. 3, β is cumulative area of the particulates with the size less than normalized size (α). Thus, β can be calculated utilizing Eq. 5.

$$\beta = 1 - \frac{Ac\,(area\ of\ \text{particles greater than normalized}\ diameter, \alpha)}{A_t\,(\text{total area of the carpet or total area resulted from particles projection})} \quad (5)$$

To obey Eq. 2 and find Sierpinski's carpet 2D-fractal dimension (D_2) representing a given size distribution, we need to know A_c and A_t which are difficult to calculate from the fractal point of view. To solve this issue, one can assume a certain size distribution given on the mass basis, and according to fractal geometry area (A_i) resulted from projection of all particles found in the i^{th} size fraction and use equation:

$$A_t=\theta\,(N\,A_{oi}) \tag{6}$$

where N = number of particles in that size fraction, and A_{oi} is the area of one particle in this size fraction based on average diameter d, and θ is the packing factor. Considering the particles irregularity and taking into account the fractal dimension definition, A_{oi} can be calculated as:



a b c

Fig. 3. Sierpinski's carpet 2D-fractal (a). Sierpinski's carpet after "shaking" (b), size distribution curve (c) based on Fig.3b

$$A_{oi}=\delta\,d_i^{D2} \tag{7}$$

where δ = shape factor or irregularity factor which can be also named fractal prefactor, and D_2 is the area-based fractal dimension. At the same time the number of particles in a given size fraction (N) can be given by the following equation:

$$N=m_i/m_{oi}=m_i/(v_{0i}\,\rho)=m_i/(A_{oi}\,d_i^{D2/2}\,\rho) \tag{8}$$

where m_i is the material mass fraction found in the i^{th} size class, m_{oi} and v_{oi} are the mass and volume of one particle located in the i^{th} size class, respectively. In Eq. 8, ρ is the bulk density of the material under analyses (assumed to be constant for all size classes). Inserting Eqs 7 and 8 in Eq. 6, one can get the projected area resulted from the i^{th} size fraction:

$$Ai=\frac{\theta}{\rho}\left(\frac{m_i}{d_i^{D2/2}}\right) \tag{9}$$

Thus, the total projected area (A_t) for the whole size distribution can be calculated according to the general form (given by Eq. 10) assuming the packing factor to be constant for all the size classes:

$$A_t=\frac{\theta}{\rho}\left(\frac{m_1}{d_1^{D2/2}}+\frac{m_2}{d_2^{D2/2}}+\frac{m_3}{d_3^{D2/2}}+\ldots\ldots\frac{m_n}{d_n^{D2/2}}\right) \tag{10}$$

and the cumulative area greater than normalized α can be given according to the following equations:

$$A_{c1}\% = \left[\frac{\frac{m_1}{d_1^{D_2/2}}.100}{\frac{m_1}{d_1^{D_2/2}} + \frac{m_2}{d_2^{D_2/2}} + \frac{m_3}{d_3^{D_2/2}} + \ldots\ldots \frac{m_n}{d_n^{D_2/2}}} \right] \tag{11}$$

$$A_{c2}\% = \left[\frac{(\frac{m_1}{d_1^{D_2/2}} + \frac{m_2}{d_2^{D_2/2}})100}{\frac{m_1}{d_1^{D_2/2}} + \frac{m_2}{d_2^{D_2/2}} + \frac{m_3}{d_3^{D_2/2}} + \ldots\ldots \frac{m_n}{d_n^{D_2/2}}} \right] \tag{12}$$

$$A_{ci}\% = \left[\frac{(\frac{m_1}{d_1^{D_2/2}} + \frac{m_2}{d_2^{D_2/2}} + \ldots + \frac{m_i}{di^{D_2/2}})100}{\frac{m_1}{d_1^{D_2/2}} + \frac{m_2}{d_2^{D_2/2}} + \frac{m_3}{d_3^{D_2/2}} + \ldots\ldots \frac{m_n}{d_n^{D_2/2}}} \right] \tag{13}$$

Inserting Eqs 10 and 13 in Eq. 5 and taking into consideration normalized diameter ($\alpha_i = d_i/d_{max}$), one can determine the general equation for calculating β:

$$\beta = 1 - \frac{(\frac{m_1}{\alpha^{D_2/2}} + \frac{m_2}{\alpha_2^{D_2/2}} + \ldots + \frac{m_i}{\alpha i^{D_2/2}})100}{(\frac{m_1}{\alpha_1^{D_2/2}} + \frac{m_2}{\alpha_2^{D_2/2}} + \frac{m_3}{\alpha_3^{D_2/2}} + \ldots\ldots \frac{m_n}{\alpha_n^{D_2/2}})} \tag{14}$$

Equation 14 can not be solved analytically because it contains two unknowns, that is β and 2D-fractal dimension D_2. It can be solved by trial and error, and there exist only one value of D_2 which is correct for a given size distribution. It should agree with Eq. 2, rewritten in the form below:

$$\frac{\beta}{\alpha^{2-D_2}} = const \tag{15}$$

Combining results of Eq. 14 with the conditional Eq. 15 and applying trial and error numerical analyses, one can obtain the fractal dimension characterizing a certain size distribution range. The applied trial and error method optimizes the Sierpinski carpet area to fulfill the linearized size distribution by changing D_2.

TWO-DIMENSIONAL FRACTAL LINEARIZATION OF SIZE DISTRIBUTION CURVES

The size distribution curve of an Bulgarian anthracite coal (Kuzev et al., 1994, Table 5, p. 88) was plotted in Fig. 4a, and next linearized using traditional mathematical equations (Fig. 4b), 3D-fractal (Fig. 4c), and also using 2D-fractal (Fig.

4d). For determination of 2D-fractal dimension D_2 characterizing particle size distribution, the trial and error procedure proposed in this work was applied. Table 2 shows results of the last trial.

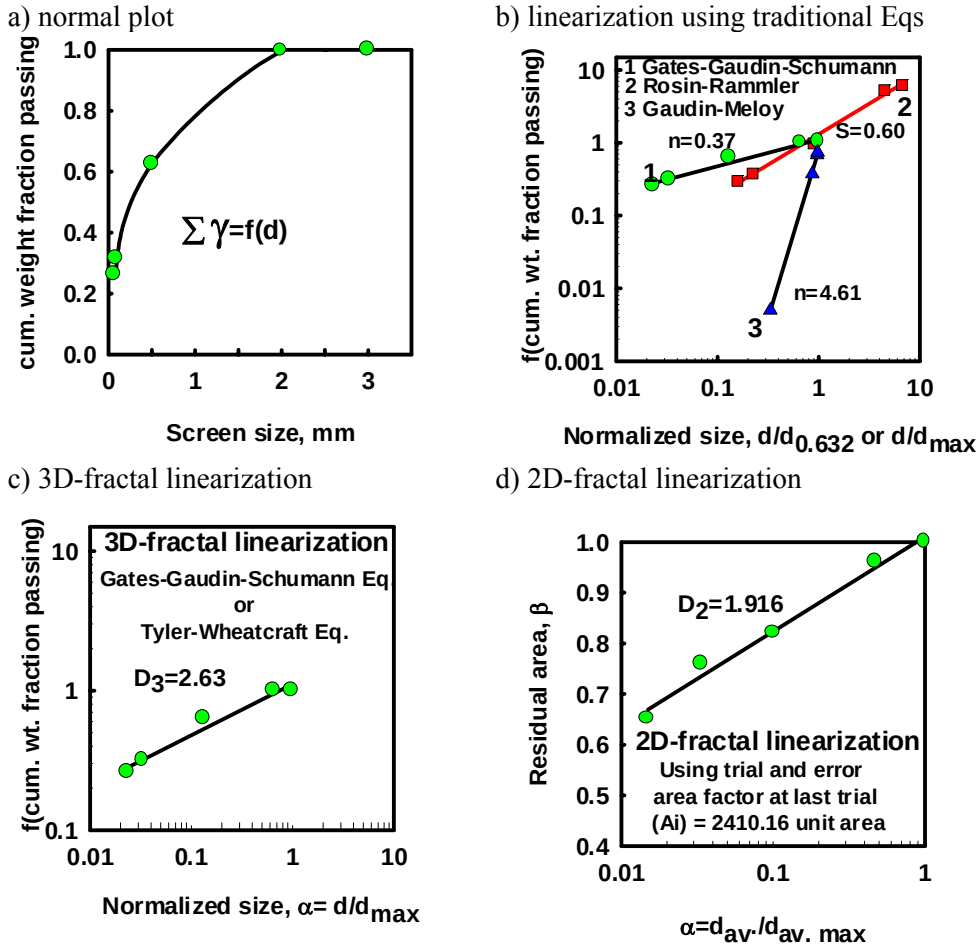


Fig. 4. Size distribution of the considered Bulgarian anthracite coal (data after Kuzev et al., 1994, Table 5, p. 88)

Figure 4d shows that single 2D-fractal dimension D_2 equal to 1.916 can represent the whole size range because the last column in Table 3 having the form $\frac{\beta}{\alpha^{2-D_2}}$ is approximately constant and equals 1. The results obtained basing on that fractal dimension show a linear character when plotting normalized size $\alpha=(d_{av}/d_{av.\,max})$ versus normalized residual area of the Sierpinski carpet on a semi-log scale (Fig. 4d). The area index (A_t) was found to be 2410 of area units.

Table 2. Trial and error results (last trial) for determination of fractal dimension (D_2) expressing size distribution of the investigated Bulgarian anthracite. D_2 is considered final when the last column in the table ($\frac{\beta}{\alpha^{2-D_2}}$) becomes constant

D_2=1.916									
d_{max}	d_{min}	$d_{av.}$	α*	(m_i),%	$A_i=m_i/\alpha^{0.5D_2}$	A_i, %	$A_{ci}=\Sigma A_i$,%	$\beta=1-A_{ci}$	β/α^{2-D_2}
3.00	2.00	2.50	1.000	0.5	0.50	$2\cdot10^{-04}$	0.0002466	1.00	0.9999
2.00	0.40	1.20	0.480	37.0	74.74	$4\cdot10^{-02}$	0.0371095	0.96	1.0004
0.40	0.10	0.25	0.100	31.0	281.42	$1\cdot10^{-01}$	0.1759061	0.82	0.9999
0.10	0.071	0.086	0.034	5.5	139.56	$7\cdot10^{-02}$	0.2447369	0.76	1.0008
0.071	0	0.036	0.014	26.0	1531.37	$8\cdot10^{-01}$	1		
				100	2410.15				

* $\alpha = d_{av} / d_{av.max}$

To check the ability of the 2D fractal to linearize a wide range of sizes, different literature size-distribution curves were considered. The results showed that frequently for wide-range size distributions a single 2D fractal is not sufficient. Usually there are two separate fractal dimensions for coarse and fine size fractions (Fig. 5).

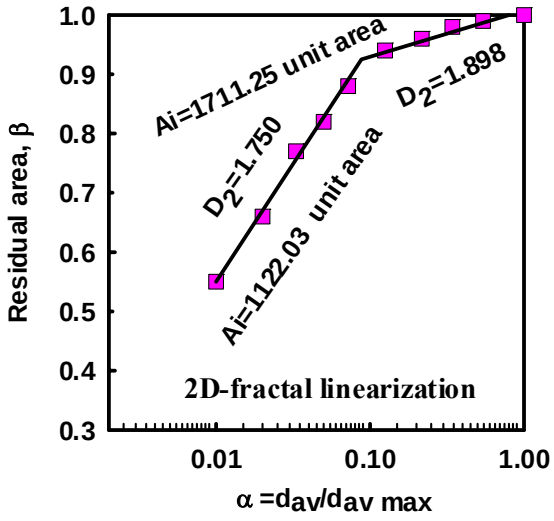


Fig. 5. Fractal linearization of a wide size-distribution range of coal (data after Ernst and Manfred, 1994, Table 1, p. 191). The coarse fractions have fractal dimension different from that representing the fine size fractions

CONCLUSIONS

The following conclusions can be drawn from the paper:

1. Fractal geometry can be used to express size distribution data in a linear form using either 2D or 3D fractals
2. Gates-Gaudin-Schumann and the 3D fractal linearization are identical in mathematical forms
3. 2D fractal linearization can be accomplished by finding an optimum background area (area on which the sample of particles is spread) by the trial and error method
4. Frequently, wide size distributions need more than one 2D or 3D fractal dimensions to express the whole range
5. There is no a universal procedure that can be followed for linearization of a given size distribution curve
6. To compare different size distribution curves on linear bases, one has to check different approaches for linearization to find the one which provides straight lines for the whole considered size range. Otherwise the whole size range should be divided into fractally linearized sub-ranges
7. 2D fractal linearization of size distribution curve is equivalent to two-adjustable parameter mathematical approach with D_2 and A_i as adjustable parameters.

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Hussin A.M. Ahmed, Drzymała J., *Linearyzacja krzywych dystrybucji za pomocą dwuwymiarowego fraktala ,* Physicochemical Problems of Mineral Processing, 39 (2005) 129-139 (w jęz. ang).

Krzywe rozkładu właściwości materiałów ziarnistych (rozmiaru, gęstości, hydrofobowości, itd.) są bardzo przydatne do charakteryzowania i kontroli wyników separacji. Często krzywe te, a zwłaszcza krzywe składu ziarnowego, są linearyzowane za pomocą różnych funkcji matematycznych takich jak Rosina-Rammlera, Gatesa-Gaudina-Schumanna, czy też Gaudina-Meloy'a. W tej pracy rozważano zastosowanie rachunku fraktalnego do linearyzacji krzywej składu ziarnowego. W pracy wykazano, że trójwymiarowa (3D) fraktalna linearyzacja składu ziarnowego jest identyczna z równaniem Gatesa-Gaudina-Schumanna. Wykazano również, że dwuwymiarowy (2D) fraktal uwzględniający powierzchnię ziaren może być użyty do linearyzacji krzywej składu ziarnowego pod warunkiem odpowiedniego doboru powierzchni, na której umieszcza się rozpatrywana próbę ziaren. Pokazano także, że w niektórych przypadkach do liniowego opisu krzywej składu ziarnowego niezbędne staje się użycie więcej niż jednego fraktala. Linearyzacja za pomocą dwuwymiarowego fraktala jest w istocie przybliżaniem składu ziarnowego za pomocą dwóch dopasowywanych parametrów, to jest wymiaru fraktalnego oraz powierzchni, na której umieszcza się próbkę.