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Kolmogoroffication of a Topological Space

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ABSTRACT

The Kolmogoroff quotient (or T_0-identification) assigns to every topological space X a canonical T_0-space $T_0(X)$ by collapsing topologically indistinguishable points. This study examines this construction from order-theoretic, closure-theoretic, and categorical perspectives. We show that the T_0 condition is exactly the antisymmetry of the quasi-order induced by the topology and that the quotient map $q_x : X \to T_0(X)$ is a canonical quasihomeomorphism. The main results establish the universal property of $T_0(X)$: Every continuous map from X into a T_0-space factors uniquely through q_x, rendering Top_o reflective in Top. For a continuous map $f : X \to Y$, we analyzed the induced morphism $T_0(f)$ and characterized the conditions under which it is injective, surjective, or a homeomorphism. Finite examples and Python-based enumerations illustrate how the quasi-order of a finite topology determines its T_0-quotient, providing a computational approach to classification of finite topological spaces.

Keywords: continuous functions, kolmogoroffication, quasihomeomorphism, reflective subcategory, separation axioms, topology, T_0-identification

1. INTRODUCTION

Let X be a topological space and write $O(X)$ (respectively, $C(X)$) for the family of open (respectively, closed) sets of X. A guiding principle in several areas of topology is that most of the information carried by X is already reflected in the order structure of $O(X)$ and in the way continuous maps act on open or closed sets. Within this perspective, Grothendieck and Dieudonné[5] introduced the notion of a quasihomeomorphism, and Yip[8] later studied related closure-theoretic conditions. Recent work keeps this circle active: Dourari et al.[9] studied topological properties preserved by quasihomeomorphisms, Denniston et al.[10] surveyed sub-Hausdorff separation axioms, and Denniston et al.[2] analyzed preservation and reflection through Kolmogorov and essentially Kolmogorov relations.

Recall that a topological space is T_0 if, for every pair of distinct points x, y, there exists an open set that contains one of them but not the other[16]. Thus, a T_0-space is the weakest kind of topological space in which distinct points are not completely topologically indistinguishable. Equivalently, the specialization preorder of the space becomes antisymmetric, and hence, a partial order. This behavior renders T_0-spaces natural in situations where topology represents information, approximation, or observation rather than metric distance. For instance, in domain theory and denotational semantics, order-theoretic topologies such as Scott topologies are used to model partial information and computational approximation[17,18]. Finite T_0 Alexandroff spaces are closely related to partially ordered sets and therefore provide a bridge between topology and combinatorial or digital models[??]. Moreover, in algebraic geometry, the Zariski topology on the spectrum of a commutative ring is typically non-Hausdorff but satisfies T_0-type distinguishability, and schemes have sober underlying topological spaces[??]. Hence T_0-spaces are not mainly used to model ordinary physical space directly; rather, they provide a minimal topological framework for distinguishing states, data, or algebraic objects in ordered, computational, finite, and algebraic structures.

At the finite and computational end, recent research on quasi covering dimension and covering dimension of finite topological spaces has used the Kolmogorov quotient and the specialization order as practical tools[11,12]. Enumeration results for finite arbitrary and T_0-topologies further connect the present construction with the combinatorics of finite spaces and transitive digraphs[13–15].

In the form adopted here, a continuous map $f : X \to Y$ is a quasihomeomorphism if the inverse-image assignment

$$\mathcal{O}(f) : \mathcal{O}(Y) \to \mathcal{O}(X), \quad V \mapsto f^{-1}(V) \tag{1}$$

is bijective. Thus, f induces an isomorphism between the lattices of open sets (Lemma 3.2), a feature that renders quasihomeomorphisms well suited for "reducing" spaces while keeping the open-set structure under strict control.

The weakest separation axiom, T_0, already captures a decisive phenomenon: It excludes the presence of distinct points that cannot be distinguished by the topology. Concretely, a space fails to be T_0 precisely when it contains *topologically indistinguishable* points, that is, distinct points $x \neq y$ having the same open neighborhoods, or equivalently,

$$\overline{\{x\}} = \overline{\{y\}} \tag{2}$$

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This indistinguishability is naturally encoded by the *quasi-order induced by the topology*:

$$x \leq y \Leftrightarrow y \in \overline{\{x\}}. \tag{3}$$

The relation $\leq$ is always a quasi-order, and the T_0 condition is exactly the antisymmetry of this quasi-order. From this aspect, passing to a T_0-space is an antisymmetrization procedure: One collapses precisely the pairs x, y satisfying both $x \leq y$ and $y \leq x$.

This notion leads to the central construction of the paper. Given a space X, define an equivalence relation by

$$x \sim y \Leftrightarrow \overline{\{x\}} = \overline{\{y\}} \tag{4}$$

and let $T_0(X) = X/\sim$ be the corresponding quotient endowed with the quotient topology; we write $q_X : X \to T_0(X)$ for the canonical surjection. The space $T_0(X)$ is T_0 by construction, and the map q_X collapses exactly the topological redundancy of X. More importantly, this quotient is not merely canonical but also rigid in a categorical sense: Every continuous map from X into a T_0-space must identify indistinguishable points, and hence, factors uniquely through q_X (Theorem 4.3). This universal property is the precise formulation of the fact that $\mathbf{Top}_0$ is reflective in **Top**.

A second goal of the present work is to understand how Kolmogoroffication behaves on morphisms. Given a continuous map $f : X \to Y$, the induced map on quotients

$$T_0(f) : T_0(X) \to T_0(Y), \quad T_0(f)([x]) = [f(x)] \tag{5}$$

encodes the part of f that survives after collapsing indistinguishability on both sides. This perspective motivates the introduction and refinement of mapping notions adapted to closures, in line with recent work on quasihomeomorphisms[9]. In addition to the weak condition ("weakly topologically one-to-one"), we introduce a stronger version ("strongly topologically one-to-one"), defined by the requirement that equality of singleton closures in the codomain forces equality of singleton closures in the domain. This strengthened notion is precisely what governs the injectivity of $T_0(f)$: We prove that $T_0(f)$ is injective if and only if f is strongly topologically one-to-one. Likewise, the surjectivity of $T_0(f)$ is characterized by a closure-theoretic surjectivity condition on f ("topologically onto").

Quasihomeomorphisms interact particularly well with this framework. The canonical projection q_X is itself a quasihomeomorphism (Proposition 4.2), and Kolmogoroffication preserves quasihomeomorphisms in the sense that $T_0(f)$ is a quasihomeomorphism if and only if f is (Proposition 4.7). Combining these ideas, we obtain a precise criterion for when Kolmogoroffication turns a map into a genuine homeomorphism: $T_0(f)$ is a homeomorphism exactly when f is a topologically onto quasihomeomorphism (Theorem 4.16). We also provide examples showing that neither hypothesis can be removed.

The paper also includes explicit computations of $T_0(X)$ for finite spaces and illustrative examples that emphasize the quasi-order induced by the topology as an effective bookkeeping device. In particular, for $X = \{1, \ldots, n\}$, we describe a computational approach that enumerates non-homeomorphic topologies, counts the T_0 and non-T_0 cases, and determines the Kolmogoroff quotient of each representative.

The construction also has a concrete modeling interpretation. If points represent possible states of a system and open sets represent observable properties, then topologically indistinguishable points are states that no available observation can separate. Passing to $T_0(X)$ removes this observational redundancy without changing the open-set lattice, producing a smaller and more informative state space for finite-space models, digital-topological data, and classification procedures based on observable features.

2. SEPARATION AXIOMS

An important family of properties in topology is the separation axioms, which describe how well points can be distinguished by the open sets of a space. These axioms form a natural hierarchy, beginning with the weakest condition T_0 and strengthening through T_D, T_1, and T_2 (Hausdorff). In this section, we introduce these axioms, describe some useful characterizations, and examine their relationships through implications and counterexamples. This discussion also prepares the ground for connecting the T_0 condition with the order-theoretic framework established previously.

Definition 2.1. Let $(X, \mathcal{T})$ be a topological space.

1. X is a *T_0-space* (or *Kolmogoroff space*) if, for every pair of distinct points $x, y \in X$, there exists an open set $O \subseteq X$ containing exactly one of them.
2. X is a *T_D-space* if, for every $x \in X$, the singleton $\{x\}$ is locally closed in X.
3. X is a *T_1-space* (or *Fréchet space*) if, for every pair of distinct points $x, y \in X$, there exists an open set $O \subseteq X$ such that $x \in O$ and $y \notin O$.
4. X is a *T_2-space* (or *Hausdorff space*) if, for every pair of distinct points $x, y \in X$, there exist disjoint open sets $U, V \subseteq X$ with $x \in U$ and $y \in V$.

Proposition 2.2. A topological space X is T_0 if and only if, for all $x, y \in X$,

$$\overline{\{x\}} = \overline{\{y\}} \;\Rightarrow\; x = y \tag{6}$$

Proof. Suppose X is T_0. Let $x, y \in X$ with $cl(\{x\}) = cl(\{y\})$. If $x \neq y$, then there exists an open set O containing one of x and y but not the other, say $x \in O$ and $y \notin O$. Since $y \in cl(\{x\}) = cl(\{y\})$, every open set containing y meets $cl(\{x\})$; but also $x \in cl(\{y\})$, so every open set containing x meets $cl(\{y\})$. In particular, O contains x and must meet $cl(\{y\})$, giving $y \in O$, which is a contradiction. Therefore, $x = y$.

Conversely, assume that $\overline{\{x\}} = \overline{\{y\}}$ implies $x = y$. Let $x \neq y$; then $\overline{\{x\}} \neq \overline{\{y\}}$. Hence, either $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$. In the first case, there exists an open set containing x but not y; in the second case, there exists an open set containing y but not x. In either case, the T_0 condition is satisfied.

Proposition 2.3. A topological space X is T_1 if and only if every singleton $\{x\}$ is closed.

Proof. Suppose X is T_1. Let $x \in X$. For each $y \neq x$, the T_1 condition provides an open set O_y with $y \in O_y$ and $x \notin O_y$. Then, $X \setminus \{x\} =? \bigcup_{y \neq x} O_y$ is open, so $\{x\}$ is closed.

Conversely, if every singleton is closed, then for any $x \neq y$, the open set $X \setminus \{x\}$ contains y but not x.

Proposition 2.4. For a topological space X and $A \subseteq X$, the set A is locally closed if and only if there exists an open set U such that

$$A = U \cap \bar{A} \tag{7}$$

In particular, the singleton $\{x\}$ is locally closed if and only if there exists an open set U such that $\{x\} = U \cap \overline{\{x\}}$.

Proof. If A is locally closed, then $A = U \cap C$ for some open set U and closed set C. Since $A \subseteq \bar{A} \subseteq C$,

$$A \subseteq U \cap \bar{A} \subseteq U \cap C = A \tag{8}$$

so $A = U \cap \bar{A}$.

Conversely, if $A = U \cap \bar{A}$ for some open U, then A is the intersection of the open set U and the closed set $\bar{A}$, and hence, locally closed.

Proposition 2.5. A topological space $(X, \mathcal{T})$ is T_0 if and only if the quasi-order $\leq$ induced by the topology is antisymmetric. Equivalently, X is T_0 precisely when $\leq$ is a partial order.

Proof. Suppose X is T_0 and $x \leq y$, $y \leq x$. Therefore, x and y belong to exactly the same open sets, which contradicts the T_0 property unless $x = y$. Hence, $\leq$ is antisymmetric.

Conversely, suppose $\leq$ is antisymmetric and $x \neq y$. Then, $x \leq y$ and $y \leq x$ cannot both hold. Therefore, there exists an open set containing one of x, y but not the other, so X is T_0.

Remark 2.6. The T_0-identification of a topological space may be viewed as the *antisymmetrization* of its quasi-order induced by the topology: One collapses exactly those pairs x, y satisfying $x \leq y$ and $y \leq x$. This order-theoretic perspective provides the conceptual foundation for the construction introduced in the next section.

For more information on T_0-spaces and T_0-identification, see Denniston et al.[2].

Proposition 2.7. For any topological space X, the following implications hold:

$$T_2 \;\Rightarrow\; T_1 \;\Rightarrow\; T_D \;\Rightarrow\; T_0. \tag{9}$$

Proof. $T_2 \Rightarrow T_1$. Let $x \neq y$. The Hausdorff property provides disjoint open sets $U \ni x$ and $V \ni y$. In particular, U contains x but not y, so X is T_1.

$T_1 \Rightarrow T_D$. If X is T_1, then by Proposition 2.3, every singleton $\{x\}$ is closed, and since $\{x\} = X \cap \{x\}$, each singleton is closed. Thus, X is T_D.

$T_D \Rightarrow T_0$. Assume every singleton is locally closed. Let $x \neq y$. By Proposition 2.4, $\{x\} = U \cap \overline{\{x\}}$ for some open set U. If $y \notin U$, then U contains x but not y. If $y \in U$, then $y \notin \overline{\{x\}}$ (otherwise $y \in U \cap \overline{\{x\}} = \{x\}$, contradicting $x \neq y$). Hence there exists an open set containing y but not x. In either case, the T_0 condition holds.

The aforementioned implications are strict; none of the reverse implications hold in general.

Example 2.8 (T_1 but not T_2). Let X be an infinite set with the cofinite topology $\mathcal{CF} = \{U \subseteq X : X \setminus U \text{ is finite}\} \cup \{\emptyset\}$. Then, X is T_1: For $x \neq y$, the set $X \setminus \{y\}$ is open, contains x, and excludes y. However, X is not Hausdorff: Any two nonempty open sets U and V satisfy $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$, which is a finite union of finite sets, hence finite. Since X is infinite, $U \cap V \neq \emptyset$, so no two distinct points can be separated by disjoint open sets.

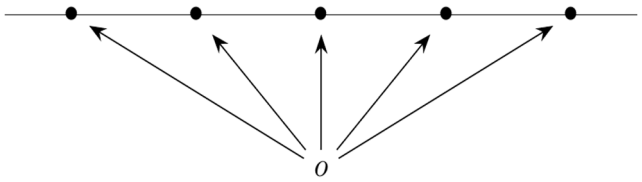


Figure 1. X is T_0. For any $p \in L$, the open set $X \setminus \{p\}$ contains o but not p, so o and p are distinguishable. For distinct $p, q \in L$, the open set $X \setminus \{p\}$ contains q but not p (see Figures 2 and 3). Hence, X is T_0

Example 2.9 (T_D but not T_1). Let $S = \{0, 1\}$ with the Sierpiński topology $\mathcal{T} = \{\emptyset, \{0\}, \{0, 1\}\}$. Then, S is T_D: $\{0\}$ is open (hence locally closed), and $\{1\}$ is closed (hence locally closed). However, S is not T_1 because $\{0\}$ is not closed.

Example 2.10 (T_0 but not T_D). Let $L = \{(t, 1) : t \in \mathbb{R}\} \subset \mathbb{R}^2$ and set $X = \{o\} \cup L$, where $o = (0, 0)$. Define a topology $\mathcal{T}$ on X by

$$\mathcal{T} = \{?, X\} \;\cup\; \{\, X \setminus F : F \subseteq L \text{ is finite} \,\}. \tag{10}$$

Thus, every nontrivial open set contains o and all but finitely many points of L.

Quasi-order induced by the topology. The nonempty closed sets are X and the finite subsets of L. The point o is dense: $\overline{\{o\}} = X$, since every nonempty open set $U = X \setminus F$ ($F \subseteq L$ finite) contains o. For $p \in L$, however, $\overline{\{p\}} = \{p\}$. Since $x \leq y \Leftrightarrow y \in \overline{\{x\}}$, we have $o \leq y$ for every $y \in X$, while distinct points $p, q \in L$ are incomparable. The Hasse diagram shown in Fig. 1 has o as a bottom element, with the points of L pairwise incomparable.

X is not T_D. For $p \in L$, the singleton $\{p\}$ is closed, hence locally closed. However, $\{o\}$ is not locally closed. Indeed, if $\{o\} = U \cap \overline{\{o\}}$ for some open U, then $\{o\} = U \cap X = U$. However, $\{o\}$ is not open: Every nontrivial open set contains o together with all but finitely many points of L, so no open set is a singleton. Thus, X is not T_D (Fig. 2 and 3).

3. QUASIHOMEOMORPHISMS

In this section, we introduce the notion of a quasihomeomorphism, a concept due to Grothendieck and Dieudonne[5] and independently studied by Yip[8]. A quasihomeomorphism is a continuous map that induces a bijection on open sets, thereby preserving an important aspect of the topological structure even when the map is not a homeomorphism in the usual sense.

Our goal here is to present the definition together with some key properties and equivalent formulations that will serve as tools for the study of the T_0-identification described in the next section.

Definition 3.1. Let $f : X \to Y$ be a continuous map between topological spaces. We say that f is a *quasihomeomorphism* if the map

$$\begin{aligned} \mathcal{O}(f) : \mathcal{O}(Y) &\to \mathcal{O}(X) \\ V &\mapsto f^{-1}(V) \end{aligned} \tag{11}$$

is a bijection.

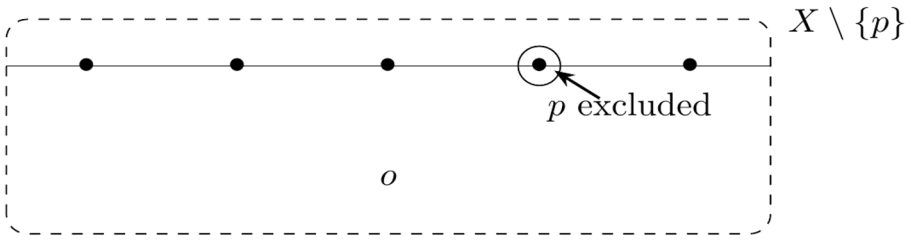


Figure 2. Open set $X\backslash\{p\}$ contains o but excludes p

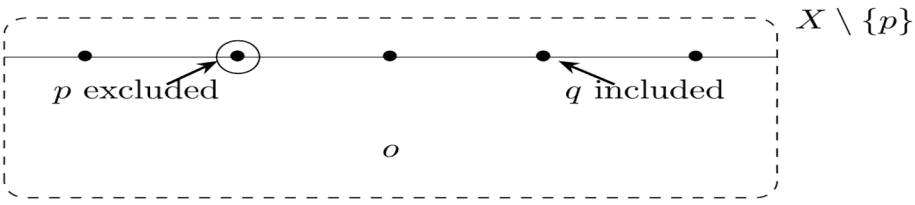


Figure 3. Open set $X\backslash\{p\}$ contains q but excludes p

By a lattice, we mean a set X together with a partial order $\leq$ such that, for all $a, b \in X$, the meet $a\Lambda b := inf(\{a, b\})$ and the join $a \bigwedge b := sup(\{a, b\})$ exist.

A lattice isomorphism $\varphi : (X, \leq) \to (Y, \preccurlyeq)$ is a bijection such that, for all $a, b \in X$, $a \leq b$ if and only if $\varphi(a) \preccurlyeq \varphi(b)$.

Lemma 3.2. Let $f : X \to Y$ be a quasihomeomorphism. Then, the map $\mathcal{O}(f) : \mathcal{O}(Y) \to \mathcal{O}(X)$ is an isomorphism of lattices.

Proof. By definition, $\mathcal{O}(f)$ is a bijection. It remains to show that it is order-preserving in both directions. Clearly, if $V_1 \subseteq V_2$ in $\mathcal{O}(Y)$, then $f^{-1}(V_1) \subseteq f^{-1}(V_2)$, so $\mathcal{O}(f)$ is order-preserving. For the converse, suppose $f^{-1}(V_1) \subseteq f^{-1}(V_2)$. Then, $f^{-1}(V_1) = f^{-1}(V_1) \cap f^{-1}(V_2) = f^{-1}(V_1 \cap V_2)$. Since $\mathcal{O}(f)$ is injective, $V_1 = V_1 \cap V_2$; hence, $V_1 \subseteq V_2$. Thus, $\mathcal{O}(f)^{-1}$ is also order-preserving, and $\mathcal{O}(f)$ is a lattice isomorphism.

Lemma 3.3. (2-out-of-3 property) Let $f : X \to Y, g : Y \to Z$, and $h = g \circ f$. If any two of f, g, h are quasihomeomorphisms, then so is the third.

Proof. Since $h = g \circ f$, we have $\mathcal{O}(h) = \mathcal{O}(f) \circ \mathcal{O}(g)$ as maps $\mathcal{O}(Z) \to \mathcal{O}(X)$. The result follows from the elementary fact that if two of three maps in a composition $\alpha = \beta \circ \gamma$ are bijections, then so is the third.

Definition 3.4. Let X be a topological space and $A \subseteq X$. We say that A is *very dense* (or *strongly dense*) in X if A meets every nonempty locally closed set of X.

Proposition 3.5. Let X be a topological space and $A \subseteq X$. Then, the inclusion $i : A?X$ is a quasihomeomorphism if and only if A is very dense in X.

Proof. ($\Rightarrow$) Suppose i is a quasihomeomorphism. Let $L = C \cap O$ be a nonempty locally closed set of X, with C closed and O open. Assume for contradiction that $A \cap L = \emptyset$. Then, $A \cap C \cap O = \emptyset$, so $A \cap O \subseteq X\backslash C$, that is, $\mathcal{O}(i)(O) \subseteq \mathcal{O}(i)(X\backslash C)$. By Lemma 3.2, $O \subseteq X\backslash C$, whence $L = O \cap C = \emptyset$, a contradiction. Hence, A is very dense.

($\Leftarrow$) Suppose A is very dense. The map $\mathcal{O}(i) : \mathcal{O}(X) \to \mathcal{O}(A)$, $U \mapsto U \cap A$, is clearly surjective (because every open set of the subspace A has the form $U \cap A$ for some open $U \subseteq X$). For injectivity, let $U_1, U_2 \in \mathcal{O}(X)$ with $U_1 \cap A = U_2 \cap A$. The set $L = U_1 \cap (X\backslash U_2)$ is locally closed, and

$$L \cap A = U_1 \cap (X\backslash U_2) \cap A = (U_1 \cap A) \backslash U_2 = (U_2 \cap A) \backslash U_2 = \emptyset. \quad (12)$$

Since A is very dense, $L = \emptyset$, so $U_1 \subseteq U_2$. By symmetry, $U_2 \subseteq U_1$; hence, $U_1 = U_2$.

Proposition 3.6. Let $f : X \to Y$ be a continuous map. The following are equivalent:

(*i*) f is a quasihomeomorphism.

(*ii*) The map $C(f) : C(Y) \to C(X), D \mapsto f^{-1}(D)$, is a bijection.

(*iii*) The topology of X is the initial topology induced by f, and $f(X)$ is very dense in Y.

Proof. (*i*) $\Leftrightarrow$ (*ii*). Taking complements establishes a canonical bijection between open and closed sets, so the bijectivity of $\mathcal{O}(f)$ is equivalent to that of $\mathcal{C}(f)$.

(*i*) $\Rightarrow$ (*iii*). Since $\mathcal{O}(f)$ is surjective, every open set in X has the form $f^{-1}(V)$ for some $V \in \mathcal{O}(Y)$, so the topology on X is the initial topology induced by f.

To show that $f(X)$ is very dense, let $L = V \cap D$ be a nonempty locally closed set of Y (V open, D closed). If $f(X) \cap L = \emptyset$, then $f^{-1}(V) \cap f^{-1}(D) = \emptyset$, so $f^{-1}(V) \subseteq f^{-1}(Y\backslash D)$. By Lemma 3.2, $V \subseteq Y\backslash D$, contradicting $L \neq \emptyset$.

(*iii*) $\Rightarrow$ (*i*). Since $f(X)$ is very dense in Y, the inclusion $i : f(X)?Y$ is a quasihomeomorphism by Proposition 3.5. Since the topology on X is the initial topology induced by f, the induced map $f_0 : X \to f(X)$ is a quasihomeomorphism. As $f = i \circ f_0$, Lemma 3.3 implies that f is a quasihomeomorphism.

In 1970, Yip introduced the following equivalent formulation of quasihomeomorphism[8].

Definition 3.7. A continuous map $f : X \to Y$ is said to be a *Y-quasihomeomorphism* if the following two conditions hold:

$$(Y_1) : f^{-1}\left(\overline{f(C)}\right) = C \text{ for every closed set } C \subseteq X. \quad (13)$$

$$(Y_2) : \overline{f\left(f^{-1}(D)\right)} = D \text{ for every closed set } D \subseteq Y. \quad (14)$$

The following examples show that conditions (Y_1) and (Y_2) are independent.

Example 3.8 ((Y_1) **without** (Y_2))**.** Let $X \subsetneq Y$ be a proper closed set of Y. The inclusion $i : X \hookrightarrow Y$ satisfies (Y_1): for any closed $C \subseteq X$, one has $i^{-1}\left(\overline{i(C)}\right) = C$. However, taking $D = Y$ gives $\overline{i\left(i^{-1}(Y)\right)} = X \neq Y$, so (Y_2) fails.

Example 3.9 ((Y_2) **without** (Y_1))**.** Let X be a discrete space with $|X| > 1$ and Y an indiscrete space. The constant map $f : X \to Y, f(x) = y_0$ satisfies (Y_2) (since the only closed sets in Y are $\emptyset$ and Y). However, for any singleton $C = \{x\}$, we obtain $f^{-1}\left(\overline{f(C)}\right) = f^{-1}(Y) = X \neq \{x\}$, so (Y_1) fails.

The two notions of quasihomeomorphism coincide.

Theorem 3.10. Let $f : X \to Y$ be a continuous map. Then, f is a quasihomeomorphism (in the sense of Definition 3.1) if and only if f is a Y-quasihomeomorphism (in the sense of Definition 3.7).

Proof. ($\Rightarrow$) Assume f is a quasihomeomorphism and let $C \subseteq X$ be closed. By Proposition 3.6 (ii), $C = f^{-1}(D)$ for some closed $D \subseteq Y$. Then, $f(C) \subseteq D$, so $\overline{f(C)} \subseteq D$, and therefore

$$C \subseteq f^{-1}\left(f(C)\right) \subseteq f^{-1}\left(\overline{f(C)}\right) \subseteq f^{-1}(D) = C \tag{15}$$

Hence, (Y_1) holds. For (Y_2), let $D \subseteq Y$ be closed and set $C = f^{-1}(D)$. By (Y_1), $C = f^{-1}\left(\overline{f(C)}\right)$. Since $\mathcal{C}(f)$ is injective, $D = \overline{f(C)} = \overline{f\left(f^{-1}(D)\right)}$.

($\Leftarrow$) Assume f satisfies (Y_1) and (Y_2). By (Y_1), the map $\mathcal{C}(f) : D \mapsto f^{-1}(D)$ is surjective: Every closed $C \subseteq X$ equals $f^{-1}\left(\overline{f(C)}\right)$. For injectivity, if $f^{-1}(D_1) = f^{-1}(D_2)$, then $\overline{f\left(f^{-1}(D_1)\right)} = \overline{f\left(f^{-1}(D_2)\right)}$, and (Y_2) gives $D_1 = D_2$. Hence, $\mathcal{C}(f)$ is bijective, so f is a quasihomeomorphism.

Proposition 3.11. A continuous map $f : X \to Y$ is a homeomorphism if and only if it is a bijective quasihomeomorphism.

Proof. Every homeomorphism is clearly a bijective quasihomeomorphism. Conversely, let f be a bijective quasihomeomorphism. It suffices to show that f is open. Let $U \in \mathcal{O}(X)$. Since f is a quasihomeomorphism, $U = f^{-1}(V)$ for a unique $V \in \mathcal{O}(Y)$. As f is bijective, $f(U) = f\left(f^{-1}(V)\right) = V$, which is open. Hence, f is a homeomorphism.

Proposition 3.12. Let $f : X \to Y$ be a continuous surjection. The following are equivalent:

(*i*) f is a quasihomeomorphism.

(*ii*) f is open, and $f^{-1}\left(f(U)\right) = U$ for every open set $U \subseteq X$.

(*iii*) f is closed, and $f^{-1}\left(f(C)\right) = C$ for every closed set $C \subseteq X$.

Proof. (*i*) $\Rightarrow$ (*ii*). Let $U \in \mathcal{O}(X)$. Since f is a quasihomeomorphism, there exists a unique $V \in \mathcal{O}(Y)$ with $f^{-1}(V) = U$. Since f is surjective, $f(U) = f\left(f^{-1}(V)\right) = V$, so $f(U)$ is open and $f^{-1}\left(f(U)\right) = f^{-1}(V) = U$.

(*ii*) $\Rightarrow$ (*iii*). Let $C \subseteq X$ be closed and set $U = X \backslash C$. By (ii), $f(U)$ is open and $f^{-1}\left(f(U)\right) = U$. Set $D = Y \backslash f(U)$; then, D is closed. Since f is surjective, $f(C) = f(X \backslash U) = Y \backslash f(U) = D$, so f is closed. Moreover, $f^{-1}\left(f(C)\right) = f^{-1}(D) = f^{-1}\left(Y \backslash f(U)\right) = X \backslash f^{-1}\left(f(U)\right) = X \backslash U = C$.

(*iii*) $\Rightarrow$ (*i*). By Proposition 3.6, it suffices to show that $\mathcal{C}(f) : D \mapsto f^{-1}(D)$ is bijective. *Injectivity:* Since f is surjective, if $f^{-1}(D_1) = f^{-1}(D_2)$, then $D_1 = f\left(f^{-1}(D_1)\right) = f\left(f^{-1}(D_2)\right) = D_2$. *Surjectivity:* for any closed $C \subseteq X$, the set $f(C)$ is closed in Y (because f is closed), and $f^{-1}\left(f(C)\right) = C$ by hypothesis.

4. THE T_0-IDENTIFICATION OF A TOPOLOGICAL SPACE

A topological space that is not T_0 may contain points that are topologically indistinguishable, namely points x, y satisfying $\overline{\{x\}} = \overline{\{y\}}$. To remove this redundancy, one associates to every topological space X a canonical T_0-space called its Kolmogoroffication or T_0-identification. This construction collapses indistinguishable points into equivalence classes, endows the quotient with the quotient topology, and yields a T_0-space that faithfully reflects the essential separation structure of X.

In this section, we define this identification, establish its basic properties, and prove that it satisfies a universal property: Every continuous map from X to a T_0-space factors uniquely through the T_0-identification. This property shows, from a categorical perspective, that the subcategory $\mathbf{Top}_0$ of T_0-spaces is reflective in **Top**.

Definition 4.1. Let $(X, \mathcal{T})$ be a topological space. Define an equivalence relation on X by

$$x \sim y \Leftrightarrow \overline{\{x\}} = \overline{\{y\}} \tag{16}$$

The Kolmogoroff quotient (or T_0-identification) assigns to every topological space X a canonical T_0-space $T_0(X)$ by collapsing topologically indistinguishable points. In this study, that construction is examined from order-theoretic, closure-theoretic, and categorical viewpoints. We show that the T_0 condition is exactly the antisymmetry of the quasi-order induced by the topology and that the quotient map $q_x : X \to T_0(X)$ is a canonical quasihomeomorphism.

$$T_0(X) = X/\sim \tag{17}$$

endowed with the quotient topology. The points of $T_0(X)$ are the equivalence classes $[x] = \left\{y \in X : \overline{\{x\}} = \overline{\{y\}}\right\}$.

Proposition 4.2. Let $q_X : X \to T_0(X)$ denote the canonical surjection $x \mapsto [x]$. Then:

1. For every open set $U \subseteq X$,

$$q_X^{-1}\left(q_X(U)\right) = U. \tag{18}$$

- In particular, q_X is an open map.

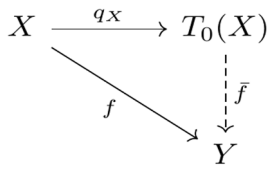
1. q_X is a quasihomeomorphism.
2. $T_0(X)$ is a T_0-space.
3. X is T_0 if and only if q_X is a homeomorphism.

Proof.

1. Let U be an open set in X. The inclusion $U \subseteq q_X^{-1}\left(q_X(U)\right)$ is immediate. For the reverse, let $x \in q_X^{-1}\left(q_X(U)\right)$; then, there exists $y \in U$ with $[x] = [y]$, that is, $\overline{\{x\}} = \overline{\{y\}}$. Since $y \in \overline{\{x\}}$ (because $y \in \overline{\{y\}} = \overline{\{x\}}$) and U is an open neighborhood of y, the set U must meet $\{x\}$, so $x \in U$. Hence, $q_X^{-1}\left(q_X(U)\right) = U$, which means $q_X(U)$ is open in the quotient topology. By definition, q_X is also continuous.
2. Since q_X is a continuous surjection satisfying $q_X^{-1}\left(q_X(U)\right) = U$ for every open U, Proposition 3.12 implies that q_X is a quasihomeomorphism.

3. Let $[x] \neq [y]$ in $T_0(X)$. Then, $\overline{\{x\}} \neq \overline{\{y\}}$, so there exists an open set $U \subseteq X$ containing one of x, y but not the other, say $x \in U$, $y \notin U$. By (i), $q_X(U)$ is open in $T_0(X)$, contains $[x]$, and does not contain $[y]$ (since $y \notin U = q_X^{-1}(q_X(U))$). Hence, $T_0(X)$ is T_0.
4. If X is T_0, then $[x] = [y]$ implies $x = y$, so q_X is bijective. Being an open, continuous bijection, q_X is a homeomorphism. Conversely, if q_X is a homeomorphism, then X is homeomorphic to the T_0-space $T_0(X)$, hence T_0.

Theorem 4.3. Let X be a topological space and Y a T_0-space. For every continuous map $f : X \to Y$, there exists a unique continuous map $\bar{f} : T_0(X) \to Y$ such that $f = \bar{f} \circ q_X$:



Proof. Define $\bar{f}([x]) = f(x)$.

Well-definedness. If $[x] = [x']$, then $\overline{\{x\}} = \overline{\{x'\}}$. By continuity of f, $f(\overline{\{x\}}) \subseteq \overline{\{f(x)\}}$ and $f(\overline{\{x'\}}) \subseteq \overline{\{f(x')\}}$. Since $x' \in \overline{\{x\}}$, we obtain $f(x') \in \overline{\{f(x)\}}$, so $\overline{\{f(x')\}} \subseteq \overline{\{f(x)\}}$. By symmetry, $\overline{\{f(x)\}} = \overline{\{f(x')\}}$. Since Y is T_0, Proposition 2.2 yields $f(x) = f(x')$.

Commutativity. By construction, $(\bar{f} \circ q_X)(x)$.

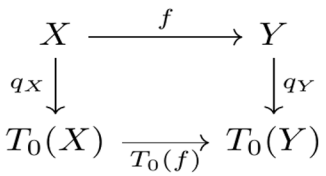
Continuity. Since q_X is a quotient map and $f = \bar{f} \circ q_X$ is continuous, $\bar{f}$ is continuous.

Uniqueness. If $g \circ q_X = f$, then $g([x]) = g(q_X(x)) = f(x) = \bar{f}([x])$ for all $[x]$.

Corollary 4.4. The subcategory $\mathbf{Top}_0$ is reflective in $\mathbf{Top}$.

Proof. For each space X, the pair $(T_0(X), q_X)$ satisfies the universal property of Theorem 4.3: $T_0(X)$ is T_0 (Proposition 4.2), and every continuous map from X to a T_0-space factors uniquely through q_X.

Proposition 4.5. Let $f : X \to Y$ be a continuous map. The map $T_0(f) : T_0(X) \to T_0(Y)$ defined by $T_0(f)([x]) = [f(x)]$ is continuous, and the diagram



commutes. The assignments $X \mapsto T_0(X)$ and $f \mapsto T_0(f)$ define a covariant functor $T_0 : \mathbf{Top} \to \mathbf{Top}_0$.

Proof. Well-definedness. If $[x] = [x']$, then $\overline{\{x\}} = \overline{\{x'\}}$. By continuity of f, $f(x') \in f(\overline{\{x\}}) \subseteq \overline{\{f(x)\}}$, so $\overline{\{f(x')\}} \subseteq \overline{\{f(x)\}}$. By symmetry, $\overline{\{f(x)\}} = \overline{\{f(x')\}}$, hence $[f(x)] = [f(x')]$.

Commutativity. $(T_0(f) \circ q_X)(x) = T_0(f)([x]) = [f(x)] = q_Y(f(x)) = (q_Y \circ f)(x)$.

Continuity. Since q_X is a quotient map and $q_Y \circ f = T_0(f) \circ q_X$ is continuous, $T_0(f)$ is continuous.

Functoriality. $T_0(1_X)([x]) = [x]$, so $T_0(1_X) = 1_{T_0(X)}$. For $g : Y \to Z$, $T_0(g \circ f)([x]) = [(g \circ f)(x)] = [g(f(x))] = T_0(g)([f(x)]) = (T_0(g) \circ T_0(f))([x])$.

Remark 4.6. The functor $T_0 : \mathbf{Top} \to \mathbf{Top}_0$ is left adjoint to the inclusion functor $I : \mathbf{Top}_0 \hookrightarrow \mathbf{Top}$.

Proposition 4.7. Let $f : X \to Y$ be a continuous map. Then, $T_0(f)$ is a quasihomeomorphism if and only if f is a quasihomeomorphism.

Proof. The commutative diagram $T_0(f) \circ q_X = q_Y \circ f$, together with the fact that q_X and q_Y are quasihomeomorphisms (Proposition 4.2), enables us to apply Lemma 3.3: $T_0(f)$ is a quasihomeomorphism if and only if $q_Y \circ f$ is, which holds if and only if f is.

We now introduce several closure-theoretic mapping notions that govern the injectivity and surjectivity of $T_0(f)$. The notions of "weakly topologically one-to-one" and "topologically onto" were defined by Belaid et al.[1]; here, we add a strengthened injectivity condition.

Definition 4.8. Let $f : X \to Y$ be a continuous map.

1. f is *weakly topologically one-to-one* if $f(x) = f(y)$ implies $\overline{\{x\}} = \overline{\{y\}}$.
2. f is *strongly topologically one-to-one* if $\overline{\{f(x)\}} = \overline{\{f(y)\}}$ implies $\overline{\{x\}} = \overline{\{y\}}$.
3. f is *topologically onto* if, for every $y \in Y$, there exists $x \in X$ such that $\overline{\{f(x)\}} = \overline{\{y\}}$.
4. f is *weakly topologically bijective* if it is both weakly topologically one-to-one and topologically onto.
5. f is *strongly topologically bijective* if it is both strongly topologically one-to-one and topologically onto.

Remark 4.9. Every injective continuous map is weakly topologically one-to-one, but an injective map need not be strongly topologically one-to-one. For example, let $X = \{a, b\}$ with the discrete topology and $Y = \{0, 1\}$ with the indiscrete topology, and define $f(a) = 0$, $f(b) = 1$. Then, f is injective, but $\overline{\{f(a)\}} = \overline{\{f(b)\}} = Y$ while $\overline{\{a\}} = \{a\} \neq \{b\} = \overline{\{b\}}$.

Proposition 4.10. Let $f : X \to Y$ be a continuous map. Then:

1. If f is strongly topologically one-to-one, then f is weakly topologically one-to-one.
2. If f is weakly topologically one-to-one and Y is T_0, then f is strongly topologically one-to-one.

Proof.

1. If $f(x) = f(y)$, then $\overline{\{f(x)\}} = \overline{\{f(y)\}}$, so the strong condition gives $\overline{\{x\}} = \overline{\{y\}}$.
2. If $\overline{\{f(x)\}} = \overline{\{f(y)\}}$ and Y is T_0, then $f(x) = f(y)$ by Proposition 2.2, and the weak condition gives $\overline{\{x\}} = \overline{\{y\}}$.

Dropping the T_0 hypothesis on Y in Proposition 4.10 causes (2) to fail.

Example 4.11. Let $X = \{a, b, c\}$ with topology $\mathcal{T} = \{\emptyset, \{c\}, X\}$, and let $Y = \{0, 1\}$ with the indiscrete topology. Define $f(a) = f(b) = 0$ and $f(c) = 1$. Then, f is weakly topologically one-to-one: If $f(x) = f(y)$, then either $\{x, y\} \subseteq \{a, b\}$ (in which case $\overline{\{x\}} = \overline{\{y\}} = X \setminus \{c\} = \{a, b\}$) or $x = y$. However, $\overline{\{f(c)\}} = \overline{\{f(a)\}} = Y$ while $\overline{\{c\}} = \{c\} \neq \{a, b\} = \overline{\{a\}}$, so f is not strongly topologically one-to-one.

The following two results characterize when $T_0(f)$ is injective and surjective, respectively.

Proposition 4.12. Let $f : X \to Y$ be a continuous map. Then, $T_0(f)$ is injective if and only if f is strongly topologically one-to-one.

Proof. Suppose $T_0(f)$ is injective. If $\overline{\{f(x)\}} = \overline{\{f(y)\}}$, then $q_Y(f(x)) = q_Y(f(y))$, so $T_0(f)([x]) = T_0(f)([y])$. Injectivity gives $[x] = [y]$, that is, $\overline{\{x\}} = \overline{\{y\}}$.

Conversely, if f is strongly topologically one-to-one and $T_0(f)([x]) = T_0(f)([y])$, then $[f(x)] = [f(y)]$, so $\overline{\{f(x)\}} = \overline{\{f(y)\}}$, whence $\overline{\{x\}} = \overline{\{y\}}$, i.e., $[x] = [y]$.

Lemma 4.13. Let $f : X \to Y$ be a continuous map. Then, $T_0(f)$ is surjective if and only if f is topologically onto.

Proof. If $T_0(f)$ is surjective, then for every $y \in Y$, there exists $x \in X$ with $T_0(f)([x]) = [y]$, i.e., $\overline{\{f(x)\}} = \overline{\{y\}}$.

Conversely, if f is topologically onto, then for every $y \in Y$, there exists $x \in X$ with $\overline{\{f(x)\}} = \overline{\{y\}}$, i.e., $T_0(f)([x]) = [y]$.

Proposition 4.14. Let $f : X \to Y$ be a quasihomeomorphism. Then:

1. f is strongly topologically one-to-one.
2. If, in addition, X is T_0, then f is injective.

Proof.

1. Let $a, b \in X$ with $cl(\{f(a)\}) = cl(\{f(b)\})$. By continuity, $f(cl(\{a\})) \subseteq cl(\{f(a)\})$ and $f(cl(\{b\})) \subseteq cl(\{f(b)\})$. Applying condition $Y1$ (Eq. (13), which holds by Theorem 3.10) to the closed sets $C = cl(\{a\})$ and $C = cl(\{b\})$:

$$\overline{\{a\}} = f^{-1}\left(\overline{f(\{a\})}\right) \text{ and } \overline{\{b\}} = f^{-1}\left(\overline{f(\{b\})}\right). \tag{19}$$

   Since $f(\overline{\{a\}}) \subseteq \overline{\{f(a)\}} = \overline{\{f(b)\}} \supseteq f(\overline{\{b\}})$, taking closures gives $\overline{f(\{a\})} = \overline{f(\{b\})}$, and therefore, $\overline{\{a\}} = \overline{\{b\}}$.
2. If X is T_0 and $f(a) = f(b)$, then $\overline{\{f(a)\}} = \overline{\{f(b)\}}$, so part (1) gives $\overline{\{a\}} = \overline{\{b\}}$, and the T_0 property forces $a = b$.

In Proposition 4.14(2), the hypothesis that X is T_0 is necessary: We cannot drop it. Let X be an indiscrete space of cardinality greater than 2 and let Y be a one-point space; the unique map $f : X \to Y$ is a quasihomeomorphism but not injective.

In general, being strongly topologically one-to-one does not imply being a quasihomeomorphism.

Example 4.15. Let $X = (0, 1)$ and $Y = [0, 1]$, both with the usual topology. The inclusion $f : X?Y$ is strongly topologically one-to-one (since Y is Hausdorff, T_0, and f is injective, so Proposition 4.10 (ii) applies). However, the closed set $D = \{0\}$ in Y satisfies $f^{-1}(D) = \emptyset$, so $\overline{f(f^{-1}(D))} = \emptyset \neq D$. Thus, (Y_2) fails, and f is not a quasihomeomorphism.

The following theorem characterizes exactly when $T_0(f)$ is a homeomorphism.

Theorem 4.16. Let $f : X \to Y$ be a continuous map. The following are equivalent:

(*i*) $T_0(f) : T_0(X) \to T_0(Y)$ is a homeomorphism.

(*ii*) f is a topologically onto quasihomeomorphism.

Proof. (*i*) $\Rightarrow$ (*ii*). If $T_0(f)$ is a homeomorphism, then it is in particular a bijective quasihomeomorphism. By Proposition 4.7, f is a quasihomeomorphism. By Lemma 4.13, f is topologically onto.

(*ii*) $\Rightarrow$ (*i*). Suppose f is a topologically onto quasihomeomorphism. By Proposition 4.7, $T_0(f)$ is a quasihomeomorphism. By Proposition 4.14, f is strongly topologically one-to-one, so $T_0(f)$ is injective (Proposition 4.12). By Lemma 4.13, $T_0(f)$ is surjective. A bijective quasihomeomorphism is a homeomorphism (Proposition 3.11).

Example 4.17 (Topologically onto but not a quasihomeomorphism). Let X be a set with cardinality greater than 1, equipped with discrete topology, and let $Y = X$ be the same underlying set equipped with indiscrete topology. If $f : X \to Y$ is the identity map, then f is a continuous bijection, hence, surjective. Therefore, it is topologically onto, but it is not a quasihomeomorphism.

Example 4.18 (Quasihomeomorphism but not topologically onto). Let X be an infinite set and let $w \notin X$. Set $Y = X \cup \{w\}$ with topology

$$\mathcal{T} = \{\emptyset, Y\} \cup \{\, Y \backslash F : F \subseteq X \text{ is finite} \,\}. \tag{20}$$

The inclusion $i : X?Y$ is a quasihomeomorphism (the open sets of X in the subspace topology are exactly $\{i^{-1}(V) : V \in \mathcal{O}(Y)\}$, and this correspondence is bijective because X is very dense in Y). However, $\overline{\{w\}} = Y$ while $\overline{\{x\}} = \{x\}$ for every $x \in X$, so no point of X has the same closure as w. Hence, i is not topologically onto, and $T_0(i)$ is not a homeomorphism.

Example 4.19. Let $X = \{0, 1, 2, 3, 4\}$ with topology

$$\mathcal{T} = \{\emptyset, \{0\}, \{0, 1, 2\}, \{0, 3, 4\}, X\}. \tag{21}$$

The quasi-order induced by the topology on $(X, \mathcal{T})$ can be represented as shown in Fig. 4.

This space is *not* T_0: The points 1 and 2 are topologically indistinguishable (both belong to the same open sets); similarly, 3 and 4 are topologically indistinguishable.

To form the T_0-identification, we identify points with equal closures: $[0] = \{0\}$, $[1] = \{1, 2\}$, $[3] = \{3, 4\}$. Setting $a = [0]$, $b = [1]$, and $c = [3]$, we obtain $T_0(X) = \{a, b, c\}$ with quotient topology

$$\mathcal{O}(T_0(X)) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\} \tag{22}$$

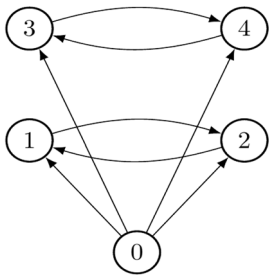
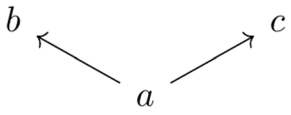


Figure 4. Quasi-order induced by the topology on X

The induced partial order on $T_0(X)$ satisfies $a \leq b$ and $a \leq c$, with b and c being incomparable, giving a V-shap+ed poset:



The space $T_0(X)$ is indeed T_0: The open sets $\{a, b\}$ and $\{a, c\}$ distinguish b from c, and every pair of distinct points is separated by some open set.

5. ALGORITHMS AND EXAMPLES

Given a finite set $X = \{1, \ldots, n\}$, we implemented a script that computes all non-homeomorphic topologies on X and the Kolmogoroff quotient of each resulting space. The script also counts the T_0 and non-T_0 topologies. We present the case $n = 3$ in detail, displaying the quasi-order on X and the induced partial order on $T_0(X)$, followed by a summary table for larger values of n.

A topology on $[n]$ is a subcollection $\tau \subseteq \mathcal{P}([n])$ containing $\emptyset$ and $[n]$ and closed under finite intersections and arbitrary unions. Two topologies are homeomorphic if one is obtained from the other by relabeling points. The algorithm enumerates all topologies and keeps one representative per equivalence class.

A topology τ is T_0 if, for all x $\neq$ y, some U $\in \tau$ contains exactly one of them. Equivalently, the quasi-order induced by the topology is a partial order, giving a bijection between T_0-topologies and partial orders on [n]. The OEIS sequence A000112 records the number of unlabeled finite posets, equivalently unlabeled T_0-topologies[9]; the companion sequence A001930 records the number of unlabeled topologies, equivalently transitive digraphs[10].

Algorithms 1 and 2 can be implemented as Python scripts to produce the following example ($n = 3$).

Algorithm 1. *Non-homeomorphic topologies on [n]*

Require Positive integer n
Ensure Set $\mathcal{R}$: one topology per homeomorphism class
Phase 1: Enumerate valid topologies
$\mathcal{T} \leftarrow \emptyset$
For all $\tau \subseteq \mathcal{P}([n])$ with $\emptyset, [n] \in \tau$ **do** Comment{2^{2^n-2} candidates}
  If τ is closed under $\cap$ and $\cup$ **then** Comment{Verify axioms (O2) and (O3)}
    $\mathcal{T} \leftarrow \mathcal{T} \cup \{\tau\}$
  End If
End For
Phase 2: Deduplicate under relabeling
$\mathcal{R} \leftarrow \emptyset$
For all $\tau \in \mathcal{T}$ **do**
  If no τ'' satisfies $\sigma(\tau) = \tau''$ for some $\sigma \in S_n$ **then** Comment{Test all $n!$ permutations}
    $\mathcal{R} \leftarrow \mathcal{R} \cup \{\tau\}$
  End If
End For
Return $\mathcal{R}$

Algorithm 2. *Non-homeomorphic* $\mathbf{T_0}$*-topologies on [n]*

Require Positive integer n
Ensure Set $\mathcal{R}$: one T_0-topology per homeomorphism class
Phase 1: Enumerate valid topologies Comment{Same as Algorithm 1, Phase 1}
$\mathcal{T}_0 \leftarrow \emptyset$
For all valid topologies τ on $[n]$ **do**
  Phase 2: Check $\mathbf{T_0}$ ***axiom***
  If for all $x \neq y$, there exists $U \in \tau$ with $|U \cap \{x, y\}| = 1$ **then** Comment{Separation test}
    $\mathcal{T}_0 \leftarrow \mathcal{T}_0 \cup \{\tau\}$
  End If
End For
Phase 3: Deduplicate under relabeling Comment{Same as Algorithm 1, Phase 2}
$\mathcal{R} \leftarrow \emptyset$
For all $\tau \in \mathcal{T}_0$ **do**
  If no τ'' satisfies $\sigma(\tau) = \tau''$ for some $\sigma \in S_n$ **then**
    $\mathcal{R} \leftarrow \mathcal{R} \cup \{\tau\}$
  End If
End For
Return $\mathcal{R}$

Example 5.1. Let $X = \{1, 2, 3\}$. There are 9 non-homeomorphic topologies on X, listed below with their T_0-identifications.

1. $\mathcal{T}_1 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

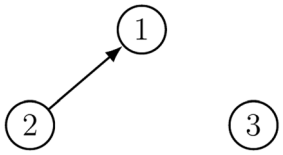


The quasi-order on X

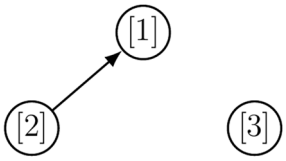


The partial order on $T_0(X)$

2. $\mathcal{T}_2 = \{\emptyset, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$

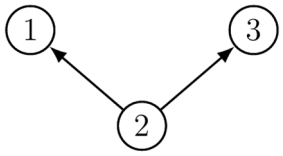


The quasi-order on X

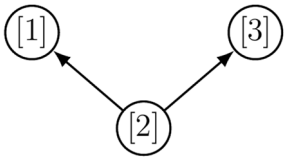


The partial order on $T_0(X)$

3. $\mathcal{T}_3 = \{\emptyset, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$

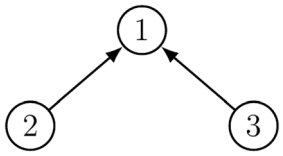


The quasi-order on X

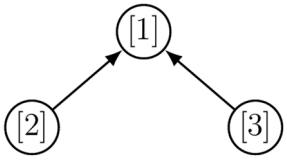


The partial order on $T_0(X)$

4. $\mathcal{T}_4 = \{\emptyset, \{2\}, \{3\}, \{2, 3\}, \{1, 2, 3\}\}$



The quasi-order on X



The partial order on $T_0(X)$

5. $\mathcal{T}_5 = \{\emptyset, \{3\}, \{2, 3\}, \{1, 2, 3\}\}$

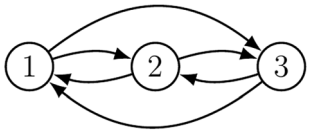


The quasi-order on X



The partial order on $T_0(X)$

6. $\mathcal{T}_6 = \{\emptyset, \{1, 2, 3\}\}$



The quasi-order on X



The partial order on $T_0(X)$

7. $\mathcal{T}_7 = \{\emptyset, \{3\}, \{1, 2\}, \{1, 2, 3\}\}$

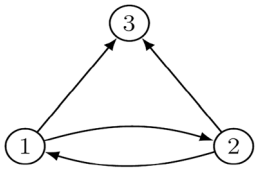


The quasi-order on X



The partial order on $T_0(X)$

8. $\mathcal{T}_8 = \{\emptyset, \{1, 2\}, \{1, 2, 3\}\}$

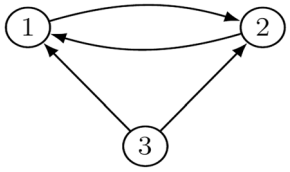


The quasi-order on X



The partial order on $T_0(X)$

9. $\mathcal{T}_9 = \{\emptyset, \{3\}, \{1,2,3\}\}$



The quasi-order on X



The partial order on $T_0(X)$

Table 1 lists the classical counts, for $n = 1, \ldots, 10$, of non-homeomorphic topologies on $\{1, \ldots, n\}$, together with the corresponding numbers of T_0 and non-T_0 topologies; these counts can be obtained easily by means of a simple Python script.

Table 1.

n	Non-homeomorphic topologies	T_0 topologies	Non-T_0 topologies
1	1	1	0
2	3	2	1
3	9	5	4
4	33	16	17
5	139	63	76
6	718	318	400
7	4535	2045	2490
8	35979	16999	18980
9	363083	183231	179852
10	4717687	2567284	2150403

6. CONCLUSION

In this study, we developed a systematic examination of the Kolmogoroffication of a topological space from order-theoretic, closure-theoretic, and categorical viewpoints. We showed that the T_0 condition is exactly the antisymmetry of the quasi-order induced by the topology, so that the T_0-identification may be regarded as the antisymmetrization of this quasi-order. This result explains why collapsing topologically indistinguishable points is the canonical method of passing from an arbitrary topological space to a T_0-space.

We also proved that the canonical surjection onto the Kolmogoroff quotient is a quasihomeomorphism and established its universal property: Every continuous map from a topological space into a T_0-space factors uniquely through the T_0-identification. Consequently, the subcategory of T_0-spaces is reflective in the category of topological spaces. In addition, we studied the induced morphism $T_0(f)$ and characterized the conditions under which it is injective, surjective, or a homeomorphism.

Furthermore, we showed that quasihomeomorphisms are preserved under Kolmogoroffication and clarified precisely when this passage yields a genuine homeomorphism. The examples show that the results are effective in concrete settings: Finite-space computations and Python-based enumeration use the specialization quasi-order as a practical tool for determining T_0-identifications and classifying finite topologies up to homeomorphism. This finding provides a natural framework linking separation axioms, quasi-orders, quasihomeomorphisms, categorical universality, and finite-state applications.

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