ACCURACY OF SEPARATION PARAMETERS RESULTING FROM ERRORS OF CHEMICAL ANALYSIS, EXPERIMENTAL RESULTS AND DATA APPROXIMATION

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Abstract: Accuracy of determination of different separation parameters and selectivity indicators depends on the error of chemical analysis of feed and separation products as well as experimental and approximation errors. In this paper different selectivity parameters were considered which formulae was based on the content of useful component in the feed, concentrate and tailing. It was shown that the impact of chemical analysis on the selectivity parameters was small and the error determined by means of partial derivative approach for a copper ore upgraded by flotation was negligible. Also experimental errors were found to be insignificant. The largest errors occurred for approximation of the upgrading data with inadequately selected selectivity indicators.

Introduction

Evaluation and approximation of upgrading results are very important elements in mineral processing. They help to find parameters which provide the best separation results for a given way of upgrading and to search for optimal conditions of processes. A knowledge of separation technological optimal points is crucial for a proper utilization of ores and it enables to minimize inevitable losses of useful components in tailings.

The principal parameters useful for evaluation of separation results are feed (α), concentrate (β) and tailing (ϑ) grades. They can be used either directly or as a combination of grades providing numerous separation parameters including recovery, yield, upgrading ratio and different selectivity parameters. For instance, the

recovery of a component in concentrate (ε) can be calculated from equation (1) (Drzymala, Ahmed, 2005):

$$\varepsilon = \frac{\alpha - \vartheta}{\beta - \vartheta} \cdot \frac{\beta}{\alpha} \cdot 100 \,, \tag{1}$$

while the recovery of others-than-considered component in the tailing (ε_r) is:

$$\varepsilon_r = \left(100 - \frac{\alpha - \vartheta}{\beta - \vartheta} \cdot 100\right) \cdot \frac{100 - \vartheta}{100 - \alpha}. \tag{2}$$

The selectively parameters of separation are usually determined by using different upgrading plots relating such parameters as grades, recoveries, yield and other parameters (Vera et al., 1999). For instance the Henry curve relates grade and yield (Neethling and Cilliers, 2008), while the Halbich curve represent grade vs. recovery (Drzymala, 2005). An especially useful upgrading curve is the Fuerstenau plot, which relates to recovery of a useful component in concentrate ε and recovery of gangue in tailing ε_r (Drzymala, 2005-2008, Drzymala et al., 2010; Brozek and Surowiak, 2010; Duchnowska and Drzymala, 2011, 2012; Foszcz, 2006; Foszcz et al., 2009; 2010; Nowak and Surowiak, 2011, 2013; Jamroz and Niedoba, 2014; Niedoba, 2013). The Fuerstenau curve provides different selectivity factors (Drzymala and Ahmed, 2005). One of them is defined as:

$$a = \frac{\varepsilon\varepsilon_r}{\varepsilon + \varepsilon_r + 100} \tag{3}$$

and its formula, expressed by means of grades of feed (α), concentrate (β), and tailing (ϑ), is:

$$a = \frac{100(\alpha-\vartheta)\beta(100\beta-100\vartheta-\alpha+\vartheta)(100-\vartheta)}{(\beta-\vartheta)[(\alpha-\vartheta)\beta 100(100-\alpha)-100(\beta-\vartheta)(100-\alpha)+\alpha(100\beta-100\vartheta+\vartheta-\alpha)(100-\vartheta)]}. \tag{4}$$

The separation parameters consist of a real value and error resulting from inaccuracy of the chemical analysis and experimental procedure as well as errors of data approximation. Therefore, the goal of this paper is to evaluate these errors using flotation results of a copper ore.

Accuracy of separation selectivity parameter *a* resulting from error of chemical analysis of products grades

According to the error analysis, the formula for the mean square error is:

$$error = \sqrt{\sum_{p=1}^{l} s_p^2 f_p^2(\bar{x}_1, \bar{x}_2..., \bar{x}_l)} \ , \qquad (5)$$

where $f_p^2(\bar{x}_1, \bar{x}_2..., \bar{x}_l)$ is partial derivative of function $f_p^2(x_1, x_2..., x_l)$ with respect to x_p, s_p^2 is standard deviation of parameter and p is from 1 to l and stands for number of variables (Wackerly and Scheaffer, 2008, Fuller, 2006). In the case of selectivity indicator a given by Eq. 4, which depends on α, β and ϑ, the error is

$$error \ a = \sqrt{\left(\frac{\partial a}{\partial \alpha}\right)^2 \cdot s_\alpha^2 + \left(\frac{\partial a}{\partial \beta}\right)^2 \cdot s_\beta^2 + \left(\frac{\partial a}{\partial \vartheta}\right)^2 \cdot s_\vartheta^2}, \qquad (6)$$

where s_α, s_β and s_ϑ are the errors of chemical analysis of the useful component in products, that is in the feed, concentrate and tailing, respectively, while $\frac{\partial a}{\partial \alpha}, \frac{\partial a}{\partial \beta}, \frac{\partial a}{\partial \vartheta}$, are partial derivatives of selectivity indicator a with respect to the grade. The formulas for $\frac{\partial a}{\partial \alpha}, \frac{\partial a}{\partial \beta}$ and $\frac{\partial a}{\partial \vartheta}$ and their calculations are given in Appendix A.

To calculate the error of selectivity indicator a determination, which results from the chemical analysis errors of α, β and ϑ, one needs to know a, α, β, ϑ, $\frac{\partial a}{\partial \alpha}, \frac{\partial a}{\partial \beta}$, $\frac{\partial a}{\partial \vartheta}$, s_α, s_β, and s_ϑ. In the case of the Kupferschiefer stratiform copper ore, typical results of separation performed on a laboratory scale is $\alpha = 2.0\%$, $\beta = 25.0\%$ and $\vartheta = 0.2\%$. For these data the selectivity indicator a, calculated by using Eq. 3, is equal to 100.60. Numerous chemical analyses of feeds and separation products showed that s values are: feed $s_\alpha = 0.06\%$, concentrate $s_\beta = 0.30\%$ and tailing $s_\vartheta = 0.03\%$. It means that the values and errors of the considered here separation results are $\alpha = 2.0 \pm 0.06\%$, $\beta = 25.0 \pm 0.30\%$ and $\vartheta = 0.2 \pm 0.03\%$. The partial derivatives of the selectivity indicator a, determined by using Eqs. 28–30 (see Appendix A) are $\frac{\partial a}{\partial \alpha} = 0.02$, $\frac{\partial a}{\partial \beta} = -0.09$, $\frac{\partial a}{\partial \vartheta} = 2.21$. After substituting these values into Eq. 6, the error of selectivity indicator a is 0.07 meaning that its value is 100.60 ± 0.07. Thus, the accuracy of a, assuming that the experiments were conducted ideally and the errors were caused only by the chemical analysis of the separation products and feed, is 100.60 ± 0.07. The error, as shown in Fig. 1, is small and should not influence interpretation and evaluation of separation data of the considered in this paper copper ore.
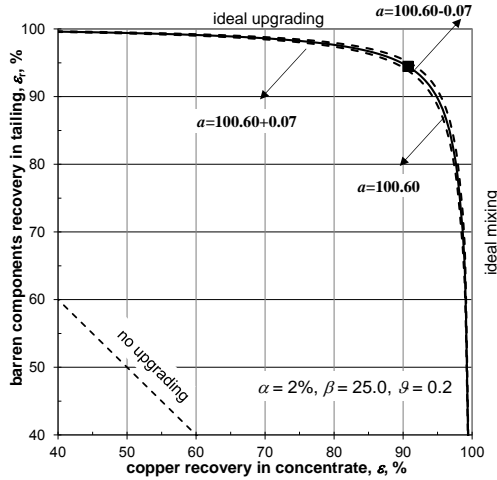
Fig. 1. Accuracy of selectivity indicator *a* resulting from feed and separation products
chemical analysis error for a single separation providing
$\alpha = 2.0\%$, $\beta = 25.0\%$, $\vartheta = 0.2\%$ for $s_\alpha = 0.06\%$, $s_\beta = 0.30\%$, $s_\vartheta = 0.03\%$

Accuracy of separation selectivity parameter *a* based on experimental errors of products grades

To find the error of *a* resulting from the experimental inaccuracy of α, β, ϑ, the data for the same experiment conducted many times are needed. It is assumed that for a given experiment, the error of the chemical analysis of feed α, concentrate β and tailing ϑ grades determination is small. To find the error of *a* resulting from the experimental error, it is convenient to use the equation based on a general formula given in Eq. 3:

$$error\ a = \sqrt{\left(\frac{\partial a}{\partial \varepsilon}\right)^2 \cdot s_\varepsilon{}^2 + \left(\frac{\partial a}{\partial \varepsilon_r}\right)^2 s_{\varepsilon_r}{}^2} \cdot \qquad (7)$$

where $\frac{\partial a}{\partial \varepsilon}$, $\frac{\partial a}{\partial \varepsilon_r}$, $s_\varepsilon{}^2$ and $s_{\varepsilon_r}{}^2$ are partial derivatives and standard deviation for ε and ε_r, respectively. The values of ε and ε_r can be calculated from α, β, ϑ using Eq. 4. The partial derivatives $\frac{\partial a}{\partial \varepsilon}$ and $\frac{\partial a}{\partial \varepsilon_r}$ can be calculated as shown in Appendix A. The partial derivatives are:

$$\frac{\partial a}{\partial \varepsilon} = \frac{\varepsilon_r{}^2 - 100 \cdot \varepsilon_r}{(\varepsilon_r - 100 + \varepsilon)^2}, \qquad (8a)$$

$$\frac{\partial a}{\partial \varepsilon_r} = \frac{100 \cdot \varepsilon + \varepsilon^2}{(\varepsilon_r - 100 + \varepsilon)^2} \ . \qquad (8b)$$

To determine the experimental error of selectivity indicator a, the considered here copper ore was subjected to separation into concentrate and tailing by flotation. The feed was a run-of-mine ore from the Polkowice Divisions of Concentrators. It was crushed and milled and subjected to organic carbon separation by flotation. The tailing was subjected to xanthate flotation of sulfides. The results of flotation, conducted four times for identical but separate samples, taking into account only the xanthate flotation process, are given in Table 1.

Table 1. Upgrading parameters for the investigated copper ore

Product	1st flotation			2nd flotation			3rd flotation			4th flotation		
	β, %	$\Sigma\varepsilon$, %	$\Sigma\varepsilon_t$, %	β, %	$\Sigma\varepsilon$, %	$\Sigma\varepsilon_t$, %	β, %	$\Sigma\varepsilon$, %	$\Sigma\varepsilon_t$, %	β, %	$\Sigma\varepsilon$, %	$\Sigma\varepsilon_t$, %
C1	40.70	62.03	98.41	46.14	63.28	98.80	46.27	55.51	98.95	42.34	60.64	98.61
C2	29.74	77.08	96.80	31.81	78.42	97.27	33.19	73.39	97.59	29.39	76.37	96.92
C3	16.75	87.44	92.35	18.06	87.10	93.58	17.00	84.77	93.25	15.28	86.20	91.97
C4	13.22	89.38	89.68	14.04	89.08	91.14	12.67	87.36	90.18	11.79	88.27	88.91
C5	8.99	90.89	83.81	8.99	90.65	85.09	8.29	89.55	83.85	8.74	89.84	84.24
C6	4.43	93.77	64.37	4.24	93.52	65.72	4.07	93.31	64.14	4.00	94.08	62.03
T	1.73	100.00	0.00	1.60	100.00	0.00	1.61	100.00	0.00	1.65	100.00	0.00
feed	$\alpha = 1.73$			$\alpha = 1.60$			$\alpha = 1.61$			$\alpha = 1.65$		

The flotation results were plotted as the upgrading Halbich curve (Fig. 2.), and next the copper recovery was read from the graph at β=30.0%. Then, the grades of the remaining components in the tailing were calculated using Eq. 1 (Table 2).
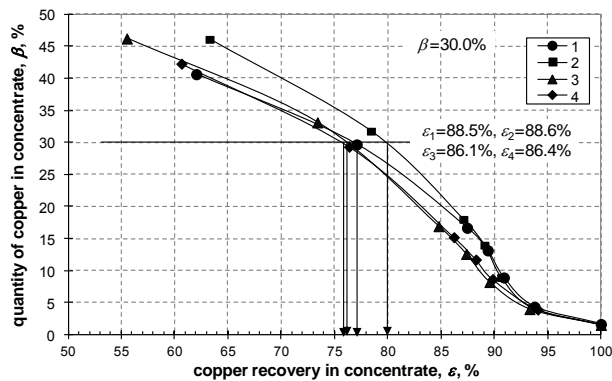

Fig. 2. The Halbich (grade – recovery) upgrading curve plotted for determination of recoveries at β=30% for four flotation tests run under the same conditions

Table 2 shows the calculated values of the derivatives and s. The calculated error of the selectivity indicator a is equal to 0.27, meaning that $a = 100.93 \pm 0.27$. Figure 3 shows graphically the results of the upgrading process and experimental error. The

experimental error for other concentrate grades can be calculated by the same procedure, also for different equations describing the Fuerstenau curve.

Since there are other equations evaluating upgrading curves based on ε and ε_r (Drzymala and Ahmed, 2005), the approximation was performed also by means of these formulae. Table 3 shows additional equations used for calculation of selectivity indicators b and c. They were determined on the basis of equations (35) and (36) presented in Appendix A. Figure 4 shows the errors resulting from the use of these indicators. Appendix A includes formulas for estimating the error of selectivity indicators b and c. Figures 3 and 4 show that the best fit, with the smallest error, is obtained for selectivity indicator b.

Table 2. Analysis of error of selectivity indicator a for a constant grade of concentrate

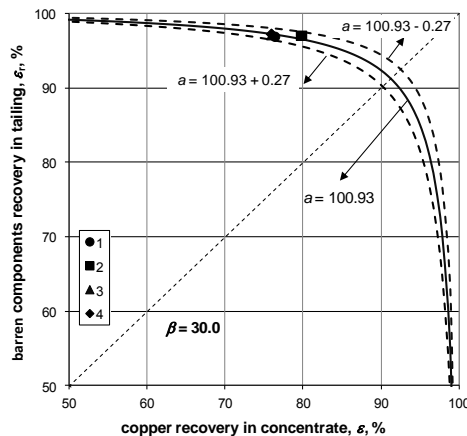| Flotation	β, %	ε, %	ε_r, %
1		76.4	96.86
2		79.8	96.97
3	30.0	75.9	97.10
4		76.1	97.02
Average		77.1	96.99
standard deviation s		1.8	0.10
average selectivity indicator a for set of all flotation ($\beta = 30.0\%$)		100.93	
$\dfrac{\partial a}{\partial \varepsilon}$ (Eq. 22) for $\varepsilon = 87.4\%$ and $\varepsilon_r = 91.7\%$		−0.05	
$\dfrac{\partial a}{\partial \varepsilon_r}$ (Eq. 23) for $\varepsilon = 87.4\%$ and $\varepsilon_r = 91.7\%$		2.49	
error a (Eq. 24)		$0.27 \rightarrow a = 100.93 \pm 0.27$	

Fig. 3. Experimental (four identical experiments) error of selectivity indicator a determination at copper content in the concentrate equal to 30%

Table 3. Selectivity indicators determined based on the Fuerstenau curve

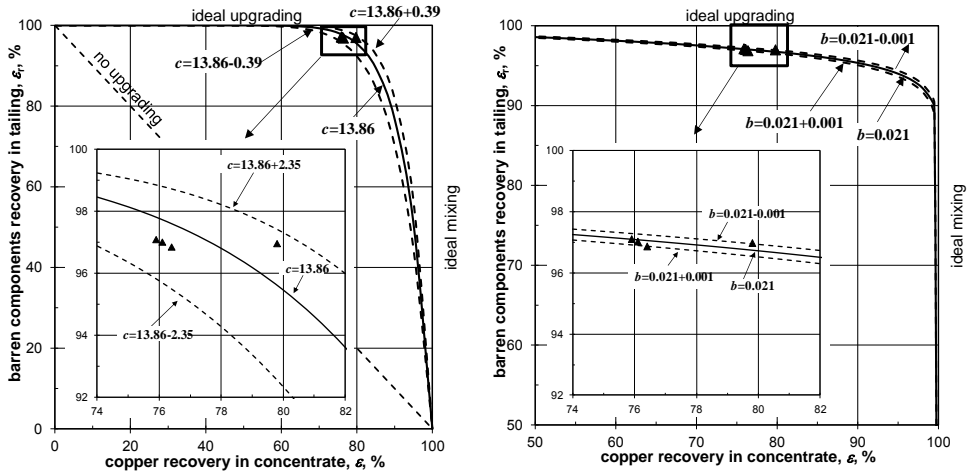| selectivity indicator	a	b	c
equation	$\varepsilon_r = a(100-\varepsilon)/(a-\varepsilon)$	$\varepsilon_r = ((100-\varepsilon)^b)/100^{(b-1)}$	$\varepsilon_r = (100^c - \varepsilon^c)/100^{(c-1)}$
ideal upgrading	100	$0, \infty$	$0, \infty$
no upgrading	∞	1	1

Fig. 4. Experimental (four identical experiments) error of selectivity indicators b
and c determination at copper content in the concentrate equal to 30%

Approximation accuracy

The experimental data can be also used to determine the error of approximation. Since the separation data can be approximated by using with different selectivity indices, the most useful are given in Table 4. The error of their approximation in the form of standard error of estimation (*SEE*) was calculated from the equation (Hair et al., 1995):

$$SEE = \sqrt{\frac{\sum_{i=1}^{n}(\varepsilon_r - \varepsilon_{rt})^2}{n-2}} \qquad (9)$$

where ε_r is barren components recovery in tailing (as-received), ε_r barren components recovery in tailing (calculated) and n stands for number of experimental points. The calculated *SEE* values are given in Table 4. It can be seen that the best approximation was obtained for selectivity parameter c (the smallest standard error of estimate).

Table 4. Approximation of the flotation tests with different selectivity indicators for Fuerstenau curve

Equation	Fitting parameter	R^2	SEE*	Fitting parameter	R^2	SEE*
	1st flotation			2nd flotation		
$\varepsilon_r = a(100-\varepsilon)/(a-\varepsilon)$	$a = 101.0634$	0.9939	2.5322	$a = 100.8004$	0.9896	3.2932
$\varepsilon_r = ((100-\varepsilon)^b)/100^{(b-1)}$ *	$b = 0.0756$	0.9446	7.9083	$b = 0.0695$	0.9437	8.0088
$\varepsilon_r = (100^c - \varepsilon^c)/100^{(c-1)}$ *	$c = 17.9810$	0.9951	2.3509	$c = 18.1697$	0.9945	2.4948
	3rd flotation			4th flotation		
$\varepsilon_r = a(100-\varepsilon)/(a-\varepsilon)$	$a = 100.9717$	0.9863	3.8038	$a = 101.0446$	0.9892	3.3633
$\varepsilon_r = ((100-\varepsilon)^b)/100^{(b-1)}$ *	$b = 0.0805$	0.9454	7.8969	$b = 0.0820$	0.9403	8.2372
$\varepsilon_r = (100^c - \varepsilon^c)/100^{(c-1)}$ *	$c = 15.8415$	0.9980	1.4988	$c = 16.6582$	0.9985	1.3132
	for date points of all flotation tests					
$\varepsilon_r = a(100-\varepsilon)/(a-\varepsilon)$	100,9612	0,9649	3,7582			
$\varepsilon_r = ((100-\varepsilon)^b)/100^{(b-1)}$ *	0,0767	0,8335	8,5213			
$\varepsilon_r = (100^c - \varepsilon^c)/100^{(c-1)}$ *	17,2063	0,9860	2,4729			

* Standard Error of Estimate, $b = \log_{\frac{100-\varepsilon}{100}} \frac{\varepsilon_r}{100}$, $c = \log_{\varepsilon/100}\left(1 - \frac{\varepsilon_r}{100}\right)$

The difference in errors of selectivity indicator c was determined for the entire set of points. The value of selectivity indicator c is 17.21 and its value is from 22.34 to 13.52 (errors taken into consideration). The value of the standard estimation error (SEE) is 2.47, with the value of R^2 equal to 0.99.
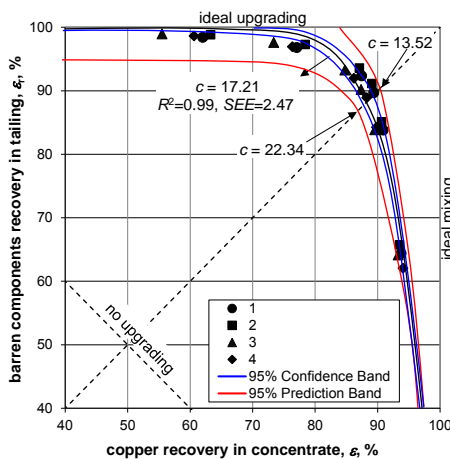

Fig. 5. Approximation of four sets of experimental data
with selectivity indicator c and it accuracy of determination

The difference in errors of selectivity indicator c was determined for the maximum position of the prediction band $[(1 - \alpha)100\%]$ determined according to a general equation for random variable y (Johnson and Wichern, 2007):

$$\hat{y} \pm t_{\alpha/2}(\text{estimated standard error of prediction}), \qquad (10)$$

where $\hat{y}$ is dependent variable (predicted), α established prediction band and t value of the t-Student statistics. The location of the confidence and prediction bands, relative to the obtained upgrading curve, was determined using SigmaPlot 11.0, while STATISTICA 9 was used to determine the confidence and prediction band at 95%. It means that the best fit line is within 95% confidence ranges (Hardle et al., 2004). The points of intersection of the prediction interval with the diagonal line joining points on the Fuerstenau graph (0,0;100,100) were also determined. Then, substituting these points to the equation for selectivity indicator c:

$$c = \log_{\varepsilon/100}\left(1 - \frac{\varepsilon_r}{100}\right) \qquad (11)$$

for ε and ε_r ((86.0,87.0); (90.0, 91.5)) (values seen in Figure 5), the range of the selectivity indicator error was obtained. Thus, value selectivity indicator c is from 13.52 to 22.34 (values calculated from Fig. 5).

To find the error of approximation of experimental data with the selectivity indicator a, similar calculations were performed and the results are presented in Fig. 6.
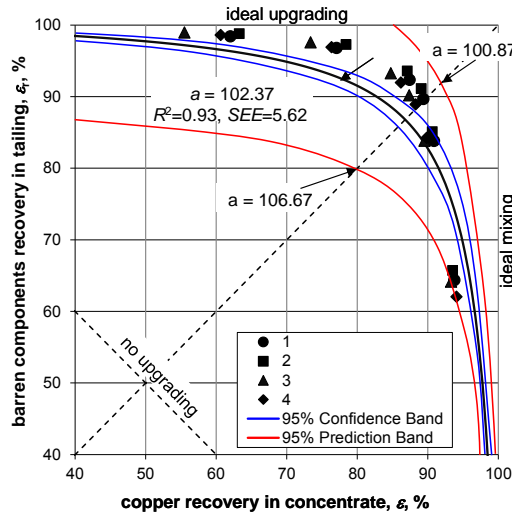

Fig. 6. Approximation of four sets of experimental data
with selectivity indicator a and it accuracy of determination

For the entire set of points the value of selectivity indicator a is 102.37, and by taking into account the standard error equal of estimate its value is from 106.67 to 100.87 (with errors taken into consideration). The value of the standard error of estimation (*SEE*) is 5.62, with the value of R^2 equal to 0.93. As in the case of selectivity indicator c, the difference in errors of selectivity indicator a value was determined by the maximum opening of the significance interval. For values ε and ε_r taken from Fig. 6 [(92.0, 91.0); (80.0; 80.0] the value of selectivity indicator a was determined basing on Eq. 3. The value of a is then from 100.87 to 106.67.

Conclusions

Determination of selectivity indicators error depends on accuracy of chemical analysis of the separation feed and process products as well as on experimental and approximation errors. For the Kupferschiefer stratiform copper ore mined by KGHM Polska Miedz S.A. the analytical errors are small and are equal to: concentrate ±0.3%, tailing ±0.03% and feed ±0.06%. The experimental errors made during laboratory flotation tests using the same ore sample and methodology are also insignificant. In the case of selectivity indicator a equal to 100.93 the error is ±0.27.

The largest errors can be made during approximation of the upgrading curves with an inappropriate selectivity indicator. Therefore, a correct determination of the selectivity indicator in the first stage of work must be based on the analysis of alignment degree of equation for the chosen selectivity indicator to the real results. The analysis shows that although the experimental error is not significant, the experiment should be conducted at least twice using the same methodology, so that the upgrading curves are of similar shape and their prediction intervals are the biggest and they overlap. Due to the fact that estimation, experimental and chemical analysis errors may sum up, a full analysis of accuracy of determination of selectivity indicator should not be simplified as in this paper, but should take into consideration the influence of all three factors simultaneously.

Acknowledgements

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Appendix A

It is convenient to calculate the derivative of a^{-1} instead of a because there is a sum in the numerator, which can be split into a sum of terms

$$a^{-1} = \frac{(\beta-\vartheta)[(\alpha-\vartheta)100\beta(100-\alpha)-100\alpha(\beta-\vartheta)(100-\alpha)+\alpha(100\beta-100\vartheta-\alpha+\vartheta)(100-\vartheta)]}{100(\alpha-\vartheta)\beta(100\beta-100\vartheta-\alpha+\vartheta)(100-\vartheta)} \tag{12}$$

or

$$a^{-1} = \frac{(\beta-\vartheta)(100-\alpha)}{(100\beta-100\vartheta-\alpha+\vartheta)(100-\vartheta)} - \frac{\alpha(\beta-\vartheta)^2(100-\alpha)}{(\alpha-\vartheta)\beta(100\beta-100\vartheta-\alpha+\vartheta)(100-\vartheta)} + \frac{\alpha(\beta-\vartheta)}{100(\alpha-\vartheta)\beta} \tag{13}$$

and briefly

$$a^{-1} = b_1 - b_2 + b_3 \tag{14}$$

For further calculations it is convenient to use logarithmic forms of term b:

$$\ln b_1 = \ln(\beta - \vartheta) + \ln(100 - \alpha) - \ln(100\beta - 100\vartheta - \alpha + \vartheta) - \ln(100 - \vartheta) \tag{15}$$

$$\ln b_2 = \ln\alpha + 2\ln(\beta-\vartheta) + \ln(100-\alpha) - \ln(\alpha-\vartheta) - \ln\beta - \ln(100\beta-100\vartheta-\alpha+\vartheta) - \ln(100-\vartheta) \tag{16}$$

$$\ln b_3 = \ln\alpha - \ln 100 - \ln(\alpha - \vartheta) - \ln\beta + \ln(\beta - \vartheta). \tag{17}$$

Since

$$[\ln(f(x))]' = \frac{1}{f(x)} \cdot f'(x) \quad \text{thus} \quad f'(x) = f(x)[\ln(f(x))]' \tag{18}$$

we get

$$\frac{\partial b_1}{\partial \alpha} = b_1 \left(\frac{-1}{100-\alpha} + \frac{1}{100\beta-100\vartheta-\alpha+\vartheta} \right) \tag{19}$$

$$\frac{\partial b_2}{\partial \alpha} = b_2 \left(\frac{1}{\alpha} - \frac{1}{100-\alpha} - \frac{1}{\alpha-\vartheta} + \frac{1}{100\beta+100\vartheta-\alpha+\vartheta} \right) \tag{20}$$

$$\frac{\partial b_3}{\partial \alpha} = b_3 \left(\frac{1}{\alpha} - \frac{1}{\alpha-\vartheta} \right) \tag{21}$$

$$\frac{\partial b_1}{\partial \beta} = b_1 \left(\frac{1}{\beta-\vartheta} - \frac{100}{100\beta-100\vartheta-\alpha+\vartheta} \right) \tag{22}$$

$$\frac{\partial b_2}{\partial \beta} = b_2 \left(\frac{2}{\beta-\vartheta} - \frac{1}{\beta} - \frac{100}{100\beta-100\vartheta-\alpha+\vartheta} \right) \tag{23}$$

$$\frac{\partial b_3}{\partial \beta} = b_3 \left(-\frac{1}{\beta} + \frac{1}{\beta-\vartheta} \right) \tag{24}$$

$$\frac{\partial b_1}{\partial \vartheta} = b_1 \left(-\frac{1}{\beta-\vartheta} + \frac{100-1}{100\beta-100\vartheta-\alpha+\vartheta} + \frac{1}{100-\vartheta} \right) \tag{25}$$

$$\frac{\partial b_2}{\partial \vartheta} = b_2 \left(-\frac{2}{\beta-\vartheta} + \frac{1}{\alpha-\vartheta} + \frac{100-1}{100\beta-100\vartheta-\alpha+\vartheta} + \frac{1}{100-\vartheta} \right) \tag{26}$$

$$\frac{\partial b_3}{\partial \vartheta} = b_3 \left(\frac{1}{\alpha-\vartheta} - \frac{1}{\beta-\vartheta} \right) \tag{27}$$

Based on equations 4-16, the partial derivatives are:

$$\frac{\partial a}{\partial \alpha} = -a^2 \left(\frac{\partial b_1}{\partial \alpha} - \frac{\partial b_2}{\partial \alpha} + \frac{\partial b_3}{\partial \alpha} \right) \tag{28}$$

$$\frac{\partial a}{\partial \beta} = -a^2 \left(\frac{\partial b_1}{\partial \beta} - \frac{\partial b_2}{\partial \beta} + \frac{\partial b_3}{\partial \beta} \right) \tag{29}$$

$$\frac{\partial a}{\partial \vartheta} = -a^2 \left(\frac{\partial b_1}{\partial \beta} - \frac{\partial b_2}{\partial \beta} + \frac{\partial b_3}{\partial \beta} \right). \tag{30}$$

Similar calculations can be performed using another formula for *a* (from the Eq. 1):

$$a = \frac{\varepsilon \varepsilon_r}{\varepsilon + \varepsilon_r - 100} \tag{31}$$

for which the partial derivatives are

$$\frac{\partial a}{\partial \varepsilon} = \frac{\varepsilon_r{}^2 - 100 \cdot \varepsilon_r}{(\varepsilon_r - 100 + \varepsilon)^2} \tag{32}$$

$$\frac{\partial a}{\partial \varepsilon_r} = \frac{100 \cdot \varepsilon + \varepsilon^2}{(\varepsilon_r - 100 + \varepsilon)^2}. \tag{33}$$

The final equation is

$$error\ a = \sqrt{\left(\frac{\partial a}{\partial \varepsilon} \right)^2 \cdot s_\varepsilon{}^2 + \left(\frac{\partial a}{\partial \varepsilon_r} \right)^2 s_{\varepsilon_r}{}^2}. \tag{34}$$

where $s_\varepsilon{}^2$ standard deviation for ε and $s_{\varepsilon_r}{}^2$ standard deviation for ε_r.

Similar calculations can be performed using another formulas for *b* and *c:*

$$b = \log_{\frac{100-\varepsilon}{100}} \frac{\varepsilon_r}{100} \tag{35}$$

$$c = \log_{\varepsilon/100} \left(1 - \frac{\varepsilon_r}{100} \right) \tag{36}$$

for which the partial derivatives are

$$\frac{\partial b}{\partial \varepsilon} = \frac{\ln\left(\dfrac{\varepsilon_r}{100}\right)}{(100 - \varepsilon)\left[\ln\left(\dfrac{100 - \varepsilon}{100}\right)\right]^2} \tag{37}$$

$$\frac{\partial b}{\partial \varepsilon_r} = \frac{1}{\varepsilon_r \ln\left(\dfrac{100 - \varepsilon}{100}\right)} \tag{38}$$

$$\frac{\partial c}{\partial \varepsilon} = -\frac{\ln\left(1 - \dfrac{\varepsilon_r}{100}\right)}{\varepsilon\left[\ln\left(\dfrac{\varepsilon}{100}\right)\right]^2} \tag{39}$$

$$\frac{\partial c}{\partial \varepsilon_r} = \frac{-1}{(100 - \varepsilon_r)\ln\left(\dfrac{\varepsilon}{100}\right)} \tag{40}$$